combined. The video cassette requires a special testing and processing of which there are only 6000 per day available. If the company makes a profit of Rs. 3/- and Rs. 5/- per audio and video cassette respectively, how many of each should be produced per day in order to maximize the profit ? **[JNTU (B. Tech.) 2003]**

40. A mining company is taking a certain kind of ore from two mines X and Y. The ore is divided into three quality groups A, B and C. Every week the company has to supply 240 tonnes of A, 160 tonnes of B and 440 tonnes of C. the cost per day for running the mine X is Rs. 3000, while if is Rs. 2000 for the mine Y. Each day, X will produce 60 tonnes of A, 20 tonnes of B and 40 tonnes of C. The corresponding figures for Y are 20, 20 and 80.

    Develop the most economical production plan by finding the number of days for which the mines X and Y work per week. **[JNTU (B. Tech.) 2003]**

OBJECTIVE QUESTIONS

Formutation

1. Mathematical model of LP problem is important because
   (a) it helps in converting the verbal description and numerical data into mathematical expression.
   (b) decision-makers prefer to work with formal models.
   (c) it captures the relevant relationship among decision factors.
   (d) it enables the use of algebraic technique.
2. Linear programming is a
   (a) constrained optimization technique. (b) technique for economic allocation of limited resources.
   (c) mathematical technique. (d) all of the above.
3. A constraint in an LP model restricts
   (a) value of objective function. (b) value of a decision variable.
   (c) use of the available resource. (d) all of the above.
4. The distinguishing feature of an LP model is
   (a) relationship among all variables is linear. (b) it has single objective function and constraints.
   (c) value of decision variables is non-negative. (d) all of the above.
5. Constraints in an LP model represents
   (a) limitations. (b) requirements.
   (c) balancing limitations and requirements. (d) all of the above.
6. Non-negativity condition is an important component of LP model because
   (a) variables value should remain under the control of decision-maker.
   (b) value of variables make sense and correspond to real-world problems.
   (c) variables are interrelated in terms of limited resources.
   (d) none of the above.
7. Before formulating a formal LP model, it is better to
   (a) express each constraint in words. (b) express the objective function in words.
   (c) decision variables are identified verbally. (d) all of the above.
8. Each constraint in an LP model is expressed as an
   (a) inequality with $\geq$ sign. (b) inequality with $\leq$ sign. (c) equation with = sign. (d) none of the above.
9. Maximization of objective function in LP model means
   (a) value occurs at allowable set of decisions. (b) highest value is chosen among allowable decisions.
   (c) neither of above (d) both (a) and (b).
10. Which of the following is not a characteristic of LP model.
    (a) alternative courses of action. (b) an objective function of maximization type.
    (c) limited amount of resources. (d) non-negativity condition on the value of decision variables.

Graphical Method

11. The feasible solution (1, 0, 1) of the system $2x_1 + 6x_2 + 2x_3 + x_4 = 3$, $6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$, $x_1, x_2, x_3, x_4 \geq 0$ is :
    (a) Basic (b) Degenerate basic feasible
    (c) Non-degenerate basic feasible (d) Not basic. **[JNTU (MCA II) 2004]**
12. The graphical method of LP problem uses
    (a) objective function equation. (b) constraint equations.
    (c) linear equations. (d) all of the above.
13. A feasible solution to an LP problem
    (a) must satisfy all of the problem's constraints simultaneously.
    (b) need not satisfy all of the constraints, only some of them.
    (c) must be a corner point of the feasible region.
    (d) must optimize the value of the objective function.

14. Which of the following statements is true with respect to the optimal solution of an LP problem
(a) every LP problem has an optimal solution.
(b) optimal solution of an LP problem always occurs at an extreme point.
(c) at optimal solution all resources are used completely.
(d) if an optimal solution exists, there will always be at least one at a corner.

15. An iso-profit line represents
(a) an infinite number of solutions all of which yield the same profit.
(b) an infinite number of solutions all of which yield the same cost.
(c) an infinite number of optimal solutions.
(d) a boundary of the feasible region.

16. If an iso-profit line yielding the optimal solution coincides with a constraint line, then
(a) the solution is unbounded.
(b) the solution is infeasible.
(c) the constraint which coincides is redundant.
(d) none of the above.

17. While plotting constraints on a graph paper, terminal points on both the axes are connected by a straight line because
(a) the resources are limited in supply.
(b) the objective function is a linear function.
(c) the constraints are linear equations or inequalities.
(d) all of the above.

18. A constraint in an LP model becomes redundant because
(a) two iso-profit lines may be parallel to each other.
(b) thé solution is unbounded.
(c) this constraint is not satisfied by the solution value.
(d) none of the above.

19. If two constraints do not intersect in the positive quadrent of the graph, then
(a) the problem is infeasible.
(b) the rolution is unbounded.
(c) one of the constraints is redundant.
(d) none of the above.

20. Constraints in LP problem are called active of they
(a) represent optimal solution.
(b) at optimality do not consume all the available resources.
(c) both of (a) and (b).
(d) none of the above.

21. The solution space (region) of an LP problem is unbounded due to :
(a) an incorrect formulation of the LP model.
(b) objective function is unbalanced.
(c) neither (a) nor (b).
(d) both (a) and (b).

22. The feasible region represented by the constraints $x_1 + x_2 \le 1$, $3x_1 + x_2 \ge 3$, $x_1 \ge 0$, $x_2 \ge 0$ of the objective function $z = x_1 + 2x_2$ is :
(a) A polygon.
(b) Unbounded set.
(c) Empty set.
(d) A singleton set.

23. In a LPP with m restrictions in n variables, the maximum number of basic feasible solutions are :
(a) ${}^nc_{m+1}$.
(b) ${}^nc_{m-2}$.
(c) ${}^nc_m$
(d) ${}^nc_{m-1}$.

Answers

1. (a)	2. (d)	3. (d)	4. (a)	5. (d)	6. (b)	7. (d)	8. (d)	9. (a)	10. (b)	11. ( )	12. (d)
13. (b)	14. (b)	15. (b)	16. (c)	17. (a)	18. (b)	19. (c)	20. (d)	21. (c)	22. (a)	23. (c)	

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Q. 1. (a) Define a convex set. Show that the intersection of two convex sets is a convex set. **[Agra 98]**

(b) Show that $S = \{ x_1, x_2, x_3 : 2x_1 - x_2 + x_3 \le 4 \} \subset R^3$ is a convex set. **[Meerut (L.P.) 90]**

2. Define convex set. Prove that the set $S = \{(x_1, x_2) \mid x_1^2 + x_2^2 \le 4\}$ is a convex set.

[Virbhadrah 2000; IGNOU (MCA II) 98]

Illustrative Examples

Example 1. *Show that* $C = \{(x_1, x_2) : 2x_1 + 3x_2 = 7\} \subset R^2$ *is a convex set.* **[Meerut (MA) 93 (P)]**

Solution. Let $\mathbf{X}, \mathbf{Y} \in C$, where $\mathbf{X} = (x_1, x_2)$, $\mathbf{Y} = (y_1, y_2)$.

The line segment joining $\mathbf{X}$ and $\mathbf{Y}$ is the set

$$W = \{\mathbf{W} : \mathbf{W} = \lambda\mathbf{X} + (1-\lambda)\mathbf{Y}, 0 \le \lambda \le 1\}$$

For some $\lambda, 0 \le \lambda \le 1$, let $\mathbf{W} = (w_1, w_2)$ be a point of set W, so that

$$w_1 = \lambda x_1 + (1-\lambda) y_1, \; w_2 = \lambda x_2 + (1-\lambda) y_2$$

Since $\mathbf{X}, \mathbf{Y} \in C$, $2x_1 + 3x_2 = 7$ and $2y_1 + 3y_2 = 7$.

But,
$$\begin{aligned} 2w_1 + 3w_2 &= 2[\lambda x_1 + (1-\lambda)y_1] + 3[\lambda x_2 + (1-\lambda)y_2] \\ &= \lambda\,[2x_1 + 3x_2] + (1-\lambda)\,[2y_1 + 3y_2] \\ &= \lambda\,.\,7 + (1-\lambda)\,.\,7 = 7 \end{aligned}$$

Therefore
$$\mathbf{W} = (w_1, w_2) \in C$$

Since $\mathbf{W}$ is any point of C, $\mathbf{X}, \mathbf{Y} \in C \Rightarrow [\mathbf{X} : \mathbf{Y}] \subset C$.

Hence C is convex.

Example 2. *For any points* $\mathbf{X}, \mathbf{Y} \in R^n$, *show that the line segment* $[\mathbf{X} : \mathbf{Y}]$ *is a convex set.*

Solution. Let $\mathbf{U}, \mathbf{V} \in [\mathbf{X} : \mathbf{Y}]$, so that

$$\left.\begin{aligned} \mathbf{U} &= t\mathbf{X} + (1-t)\,\mathbf{Y}, \; 0 \le t \le 1 \\ \text{and} \quad \mathbf{V} &= s\mathbf{X} + (1-s)\,\mathbf{Y}, \; 0 \le s \le 1 \end{aligned}\right\} \quad ...(1)$$

Now let $\mathbf{W}$ be a point of line segment $[\mathbf{U} : \mathbf{V}]$, so that

$$\mathbf{W} = \lambda\mathbf{U} + (1-\lambda)\mathbf{V}, \; 0 \le \lambda \le 1. \quad ...(2)$$

From (1) and (2), we have $\mathbf{W} = \{t\lambda + s(1-\lambda)\}\mathbf{X} + \{\lambda(1-t) + (1-\lambda)(1-s)\}\mathbf{Y}$.

If we set, $\mu = t\lambda + s(1-\lambda)$, then $1 - \mu = 1 - t\lambda - s(1-\lambda)$

$$= \lambda + (1-\lambda) - t\lambda - (1-\lambda)s = (1-t)\lambda + (1-\lambda)(1-s)$$

Since $0 \le t \le 1, 0 \le s \le 1 \Rightarrow 0 \le t\lambda + s(1-\lambda) \le 1 \Rightarrow 0 \le \mu \le 1$, therefore

$$\mathbf{W} = \mu\mathbf{X} + (1-\mu)\mathbf{Y}, 0 \le \mu \le 1 \Rightarrow \mathbf{W} \in [\mathbf{X} : \mathbf{Y}].$$

Since $\mathbf{W}$ is any point of $[\mathbf{U} : \mathbf{V}]$, we have $[\mathbf{U} : \mathbf{V}] \subset [\mathbf{X} : \mathbf{Y}]$

$$\therefore \quad \mathbf{U}, \mathbf{V} \in [\mathbf{X} : \mathbf{Y}] \Rightarrow [\mathbf{U} : \mathbf{V}] \subset [\mathbf{X} : \mathbf{Y}].$$

Hence $[\mathbf{X} : \mathbf{Y}]$ is a convex set.

Example 3. *Show that* $S = \{(x_1, x_2, x_3) : 2x_1 - x_2 + x_3 \le 4\} \subset R^3$, *is a convex set.* **[Meerut 92, 90]**

Solution. Let $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{Y} = (y_1, y_2, y_3)$ be two points of S. Then, by the given condition

$$2x_1 - x_2 + x_3 \le 4 \text{ and } 2y_1 - y_2 + y_3 \le 4 \quad ...(1)$$

Now let $\mathbf{W} = (w_1, w_2, w_3)$ be any point of $[\mathbf{X} : \mathbf{Y}]$ so that $0 \le \lambda \le 1$,

$$\therefore \quad w_1 = \lambda x_1 + (1-\lambda) y_1, \; w_2 = \lambda x_2 + (1-\lambda) y_2, \; w_3 = \lambda x_3 + (1-\lambda) y_3 \quad ...(2)$$

From (1) and (2), we have

$$\begin{aligned} 2w_1 - w_2 + w_3 &= \lambda\,(2x_1 - x_2 + x_3) + (1-\lambda)\,(2y_1 - y_2 + y_3) \\ &\le 4\lambda + 4(1-\lambda) = 4 \end{aligned}$$

$\therefore$ $\mathbf{W} = (w_1, w_2, w_3)$ is a point of S. Thus, $\mathbf{X}, \mathbf{Y} \in S \Rightarrow [\mathbf{X} : \mathbf{Y}] \subset S$

Hence S is convex.

Example 4. *Show that in* R^3, *the closed ball* $x_1^2 + x_2^2 + x_3^2 \le 1$, *is a convex set.* **[Meerut (M.Sc.) 93]**

Solution. Let $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \le 1\}$.

Also, let $\mathbf{X}, \mathbf{Y} \in S$, where $\mathbf{X} = (x_1, x_2, x_3)$, $\mathbf{Y} = (y_1, y_2, y_3)$.

Then, by the given condition, we have

$$x_1^2 + x_2^2 + x_3^2 \le 1 \text{ and } y_1^2 + y_2^2 + y_3^2 \le 1 \quad ...(1)$$

Now, for some scalar λ, $0 \le \lambda \le 1$, we have

$$\| \lambda \mathbf{x} + (1-\lambda)\mathbf{y} \|^2 = [\lambda x_1 + (1-\lambda)y_1]^2 + [\lambda x_2 + (1-\lambda)y_2]^2 + [\lambda x_3 + (1-\lambda)y_3]^2$$
$$= \lambda^2 (x_1^2 + x_2^2 + x_3^2) + (1-\lambda)^2 [y_1^2 + y_2^2 + y_3^2] + 2\lambda(1-\lambda)[x_1y_1 + x_2y_2 + x_3y_3]$$

By *Schwartz's inequality,*

$$(x_1y_1 + x_2y_2 + x_3y_3) \le \sqrt{x_1^2 + x_2^2 + x_3^2}\,\sqrt{y_1^2 + y_2^2 + y_3^2}$$

Using (1), we have

$$\|\lambda \mathbf{x} + (1-\lambda)\mathbf{y}\|^2 \le \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = [\lambda + (1-\lambda)]^2$$
$$\le 1$$

$\therefore$ $\lambda \mathbf{x} + (1-\lambda)\mathbf{y}$ is a point of S. Thus, $\mathbf{x}, \mathbf{y} \in S \Rightarrow [\mathbf{x} : \mathbf{y}] \subset S$. Hence S is convex.

4·3-1. Some Important Theorems

Theorem 4.1. *A hyperplane in R^n is a convex set.*

Proof. Let $\mathbf{cx} = z$ be a hyperplane and also let $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two points on the hyperplane. Then,

$$\mathbf{cx}_1 = z \text{ and } \mathbf{cx}_2 = z .$$

Therefore, for $0 \le \lambda \le 1$,

$\mathbf{c}[\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2] = \mathbf{c}(\lambda \mathbf{x}_1) + \mathbf{c}[(1-\lambda)\mathbf{x}_2] = \lambda(\mathbf{cx}_1) + (1-\lambda)\,\mathbf{c}\mathbf{x}_2 = \lambda z + (1-\lambda) z = z$

Hence the point $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$, for $0 \le \lambda \le 1$ lies in the hyperplane. So the hyperplane is convex.

Theorem 4.2. *The closed half spaces* $H_1 = \{\mathbf{x} \mid \mathbf{cx} \ge z\}$ and $H_2 = \{\mathbf{x} \mid \mathbf{cx} \le z\}$ *are convex sets.* **[Meerut (M.Sc) 93]**

Proof. Let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be any two points of H_1. Therefore,

$$\mathbf{c}\mathbf{x}^{(1)} \ge z \text{ and } \mathbf{cx}^{(2)} \ge z.$$

If $0 \le \lambda \le 1$, then

$$\mathbf{c}[\lambda \mathbf{x}^{(1)} + (1-\lambda)\mathbf{x}^{(2)}] = \lambda(\mathbf{cx}^{(1)} + (1-\lambda)\mathbf{cx}^{(2)} \ge \lambda z + (1-\lambda) z = z$$

Hence $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in H_1$ and $0 \le \lambda \le 1$ $\Rightarrow [\lambda \mathbf{x}^{(1)} + (1-\lambda)\mathbf{x}^{(2)}] \in H_1$. So H_1 is convex.

Similarly, if $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in H_2$, $0 \le \lambda \le 1$, then replacing the inequality sign '$\ge$' by '$\le$' in above, it is true that

$$[\lambda \mathbf{x}^{(1)} + (1-\lambda)\mathbf{x}^{(2)}] \in H_2$$

So H_2 is also convex.

Corollary. *The open half spaces* : $\{\mathbf{x} \mid \mathbf{cx} > z\}$ and $\{\mathbf{x} \mid \mathbf{cx} < z\}$ *are convex sets.*

Proof. Exercise for the student.

Theorem 4.3. **(a)** *The intersection of two convex sets is also a convex set.* **[Meerut 90]**

(b) *Intersection of any finite number of convex sets is also a convex set.*

Proof. (a) Let C_1 and C_2 be two convex sets and also let $C = C_1 \cap C_2$.

To show that C is convex.

Let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C$ and $S = \{\mathbf{x} \mid \mathbf{x} = \lambda \mathbf{x}^{(1)} + (1-\lambda)\mathbf{x}^{(2)}, 0 \le \lambda \le 1\}$

Now, $\quad \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C \Rightarrow \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C_1$ (C_1 being convex)

Also, $\quad \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C \Rightarrow \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C_2$ (C_2 being convex)

Therefore, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C \Rightarrow S \subset C_1$ and $S \subset C_2 \Rightarrow S \subset C_1 \cap C_2 \Rightarrow S \subset C$.

Hence C is convex.

(b) Let $C_1, C_2, \ldots, C_n$ be n convex sets and $C = C_1 \cap C_2 \cap \ldots \cap C_n$.

But, $\quad \mathbf{x}_1 \in C_1 \cap C_2 \cap \ldots \cap C_n \Rightarrow \mathbf{x}_1 \in C_i$, for all $i = 1, 2, \ldots, n$

and $\quad \mathbf{x}_2 \in C_1 \cap C_2 \cap \ldots \cap C_n \Rightarrow \mathbf{x}_2 \in C_i$, for all $i = 1, 2, \ldots, n$.

Since C_i is convex set for all $i = 1, 2, \ldots, n$,

$\therefore \quad \mathbf{x}_1, \mathbf{x}_2 \in C_i \Rightarrow \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C_i$, for all $i = 1, 2, \ldots, n$, where $0 \le \lambda \le 1$

$\Rightarrow \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \subseteq C_1 \cap C_2 \cap \ldots \cap C_n, \quad 0 \le \lambda \le 1.$

That is, $\mathbf{x}_1 \in C_1 \cap C_2 \cap \ldots \cap C_n$ and $\mathbf{x}_2 \in C_1 \cap C_2 \cap \ldots \cap C_n$

$\Rightarrow \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \subseteq C_1 \cap C_2 \cap \ldots \cap C_n$, $0 \le \lambda \le 1$.

3. Explain the convex combination of vectors.
4. What is meant by convex combination of vectors ? Prove that the set of all convex combinations of linearly independent vectors is a convex set.
5. Prove that any point on the line segment joining two points in R^n can be expressed as a convex combination of two points. Examine the converse for validity.

4·6. CONVEX–HULL

Definition. *The convex hull* C(*X*) *of any given set of points X is the set of all convex combinations of sets of points from X.*

In other words, the intersection of all convex sets, containing $X \subset R^n$, is called the ***convex hull*** of X and is danoted by $< X >$.

Symbolically, if $X \subset R^n$, then $< X > = \cap W_i$ where for each $W_i \supset X$ and W_i is a convex set.

Since the intersection of the members of any family of convex sets is convex, it follows that $< X >$, the convex hull of X, is a convex set.

Now for any set $X \subset R^n$, we have :

(i) $< X >$ is a convex set, $X \subset < X >$, and

(ii) if $W \supset X$ be a convex set, then $< X > \subset W$.

Thus the convex-hull of a set $X \subset R^n$ is the smallest convex set containing X.

For example :

(i) If X is just the eight vertices of a cube, then the convex hull C(*X*) is the whole cube.

(ii) If X is the boundary of a circle, then C(*X*) is the whole circle.

(iii) Let the five points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)}$ and $\mathbf{x}^{(5)}$ be given in two–dimensional space as shown in Fig. 4·4. Then, the dotted lines represent the boundaries of the convex hull for these five-points.

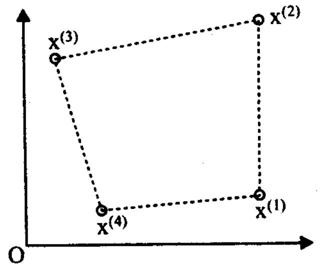

Fig. 4.4

Furthermore, the *edge* of a convex hull is defined as the line joining two of its adjacent extreme points. It should be noted that the two extreme points $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are adjacent while $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(3)}$ are not.

Theorem 4.6. *If V is any finite subset of vectors in R^n, then the **convex-hull** of V is the set of all convex combinations of vectors in V.* **[IAS (Main) 89]**

Proof. Let V be a finite subset of vectors in R^n and $< V >$ be its convex-hull.

Also, let S be the set of all convex combinations of vectors in V. Then, clearly S is a convex set containing V, and hence $< V > \subset S$.

Again $< V >$ contains V. Now show that this implies $S \subset < V >$. To prove this, finite induction on the number, m, of vectors in V will be used.

For $m = 2$, let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ be two vectors in V. Then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ are contained in $< V >$ which is a convex set. Hence $\mathbf{x} = \lambda \mathbf{x}^{(1)} + (1 - \lambda)\, \mathbf{x}^{(2)}$, $0 \le \lambda \le 1$, also lies in $< V >$.

Therefore, all convex combinations of $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ lie in $< V >$.

Now assume that, for any positive integer m, the result is true for a set of at most $m - 1$ vectors.

Consider the set $V = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)}\}$. Then $< V >$ is a convex set containing V and in particular $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k-1)}\}$. Let

$$X = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^{m-1} \lambda_i \mathbf{x}^{(i)}, \quad \lambda_i \ge 0, \quad \sum_{i=1}^{m-1} \lambda_i = 1 \right\}$$

But, by induction hypothesis $X \subset < V >$.

Also, $< V >$ contains X, as well as $\mathbf{x}^{(m)}$. Therefore, $< V >$ contains all line segments joining $\mathbf{x}^{(m)}$ to $\mathbf{x}^{(1)}$, *i.e.*

$$\mathbf{x} = \mu\, \mathbf{x}^{(m)} + (1-\mu) \sum_{i=1}^{m-1} \lambda_i \mathbf{x}^{(i)}, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{m-1} \lambda_i = 1, 0 \leq \mu \leq 1$$

is a point in $< V >$, which implies that $\mathbf{x} = \sum_{i=1}^{m} \beta_i \mathbf{x}^{(i)}$, where $\beta_i = (1-\mu)\lambda_i$, $\beta_m = \mu$ for $i = 1, 2, \ldots, m-1$.

Since $\lambda_i \geq 0$ for each i and $0 \leq \mu < 1$, $\beta_i \geq 0$ for $i = 1, 2, \ldots\, m$.

Also,
$$\sum_{i=1}^{m} \beta_i = \mu + (1-\mu) \sum_{i=1}^{m-1} \lambda_i = 1 .$$

Therefore, $\mathbf{x} = \sum_{i=1}^{m} \beta_i \mathbf{x}^{(i)}$ is a convex combination of the vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ and lies in $< V >$. Thus $S \subset < V >$.

Hence $< V > = S$.

Example 7. *Let $A = \{(\mathbf{X}, \mathbf{Y}) \subset R^n$, then show that $< A > = [\mathbf{X} : \mathbf{Y}]$.*

Solution. Since the line segment $[\mathbf{X} : \mathbf{Y}]$ is a convex set and $\mathbf{X}, \mathbf{Y} \in [\mathbf{X} : \mathbf{Y}]$, so

$[\mathbf{X} : \mathbf{Y}]$ is a convex set containing A. ....(1)

If $W \subset R^n$ be a convex set containing A, then

$\mathbf{X}, \mathbf{Y} \in A \Rightarrow \mathbf{X}, \mathbf{Y} \in W \Rightarrow [\mathbf{X} : \mathbf{Y}] \subset W$ ...(2)

From (1) and (2), we have $< A > = [\mathbf{X} : \mathbf{Y}]$.

Q. 1. Define convex-hull of a set. Prove that the convex-hull of a finite number of points is a convex set. **[Delhi BSc (Maths) 93]**

2. Obtain the convex hull of the boundary of a circle.

4.7. CONVEX POLYHEDRON, CONVEX CONE, SIMPLEX AND CONVEX FUNCTION

Definition. *If the set* **X** *consists of a finite number of points, the convex-hull of* **X** *is called a* ***convex polyhedron.***

Alternatively, *the set of all convex combinations of a finite number of points is called the convex polyhedron spanned by these points.*

For example, convex hull of eight vertices of a cube is a convex polyhedron.

Convex cone. *A non-empty subset $C \subset R^n$ is called a* ***cone*** *if for each* $\mathbf{X} \in C$, and $\lambda \geq 0$, *the vector* $\lambda\mathbf{X}$ *is also in C.*

A cone is called a ***convex cone*** *if it is a convex set.*

For example, if A be an $m \times n$ matrix then the set of n vectors $\mathbf{X}$ satisfying the constraint $\mathbf{AX} \geq \mathbf{0}$ is a convex cone in R^n. It is a cone because if $\mathbf{A}\,\mathbf{X} \geq \mathbf{0}$, then $\mathbf{A}\,(\lambda\mathbf{X}) \geq \mathbf{0}$ for $\lambda \geq 0$.

It is convex because if $\mathbf{A}\,\mathbf{X}^{(1)} \geq \mathbf{0}$ and $\mathbf{A}\,\mathbf{X}^{(2)} \geq \mathbf{0}$, then $\mathbf{A}\,[\lambda\,\mathbf{X}^{(1)} + (1-\lambda)\,\mathbf{X}^{(2)}] \geq \mathbf{0}$.

Simplex. *A simplex is an n-dimensional convex polyhedron having exactly n + 1 vertices.*

For example, a simplex in zero-dimension is a point ; in one-dimension it is a line ; in two–dimension it is a triangle ; and in three-dimension it is a tetrahedron.

Convex Function. *A function $f(\mathbf{x})$ is said to be strictly convex at* $\mathbf{x}$ *if for any two other distinct points* $\mathbf{x}_1$ and $\mathbf{x}_2$, $f\{\lambda \mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2\} < \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)$, *where* $0 < \lambda < 1$.

On the other hand, a function $f(\mathbf{x})$ is strictly concave if $-f(\mathbf{x})$ is strictly convex. We are now in a position to prove the result which is very important.

Q. 1. In your words explain the following terms :
(i) Polytope, (ii) Hyperplane, (iii) Simplex, (iv) Convex cone. **[Meerut (MA) 93]**

2. Define : Hyperplane, Simplex, and Convex polydedron.

3. What is meant by convex polyhedron ?

4. For a convex cone, show that the positive sum of any two vectors in the cone is also in the cone.

5. Define a convex polygon. Is every convex set in R^n a polygon also ?

6. Draw a convex polygon and a non-convex polygon.

7. Define a convex function and prove that the sum of two convex functions is also a convex function. **[Virbhadrah 2000]**

Maximize $z = 0{\cdot}50x_2 - 0{\cdot}10x_1 + 0x_3 + 0x_4$, subject to

$$\left.\begin{aligned} 2x_1 + 5x_2 + x_3 &= 80 \\ x_1 + x_2 + x_4 &= 20 \end{aligned}\right\}, \text{ and } x_1, x_2, x_3, x_4 \geq 0. \quad \text{...(4.4)}$$

For a system of m equations in n variables (when $n > m$) a solution in which at least $(n - m)$ of the variables have the value zero is a *vertex*. This solution is called a ***basic solution.***

To determine the basic solutions of the system (4·4) put $n - m$ variables equal to zero at a time and solve the resulting system of equations to obtain a basic solution. Here $n = 4$ (the number of variables) and $m = 2$) the number of equations). So $n - m = 2$ variables should be zero at a time. This can be done in $^4C_2 = 6$ number of ways. Hence at the most there will be 6 basic solutions which can be obtained as follows :

Set 1. *When* $x_1 = x_2 = 0$, system (4·4) gives the basic solution $x_3 = 80, x_4 = 20$.

Set 2. *When* $x_3 = x_4 = 0$, system (4·4) becomes : $2x_1 + 5x_2 = 80$ and $x_1 + x_2 = 20$ which on solving gives the basic solution $x_1 = 20/3, x_2 = 40/3$.

Set 3. *When* $x_2 = x_3 = 0$, the system (4·4) gives the basic solution : $x_1 = 40, x_4 = -20$, which is *infeasible* also.

Set 4. *When* $x_2 = x_4 = 0$, the system (4·4) gives the basic solution : $x_1 = 20, x_3 = 40$.

Set 5. *When* $x_1 = x_4 = 0$, the system (4·4) gives the basic solution : $x_2 = 20, x_3 = -20$, which is also *infeasible.*

Set 6. *When* $x_1 = x_3 = 0$, the system (4·4) gives the basic solution : $x_2 = 16, x_4 = 4$.

Substituting the values of basic variables in the objective function, the corresponding values of z are obtained as below :

Set	*Basic Solution* (x_1, x_2, x_3, x_4)	*Objective Function* $z = -0{\cdot}10x_1 + 0{\cdot}50x_2 + 0x_3 + 0x_4$
(1)	(0, 0, 80, 20)	0
(2)	(20/3, 40/3, 0, 0)	6
(3)*	(40, 0, 0, – 20)	Infeasible
(4)	(20, 0, 40, 0)	– 2
(5)*	(0, 20, – 20, 0)	Infeasible
(6)	(0, 16, 0, 4)	8

Since solution-sets (3)* and (5)* yield a negative coordinate, each contradicting thereby the non-negativity constraints, these are infeasible and so are dropped from the consideration. The optimum solution thus obtained is : $x_1 = 0, x_2 = 16$, max $z = 8$ as obtained earlier also by graphical method.

Further, it is important to observe that four basic feasible solution sets (1), (2), (4) and (6) exactly coincide with the corner points O, E, C and B of the feasible region (see Fig.3·7) respectively, and one of these corner points gives the optimal solution.

Extreme point theorem also states that *an optimal solution to an LP problem occurs at one of the vertices of the feasible region.*

Since the vertices of the feasible region are corresponding to the *basic feasible solutions,* the objective function is optimal at least at one of the basic solutions. Some of the vertices may be infeasible which are dropped from consideration.

Disadvantages of analytical method :

1. In LP problems in which m and n are large, solution of various sets of simultaneous equations become extremely difficult and time consuming.
2. Some of the sets give infeasible solutions also. There should be some technique to detect all such sets and not solve them at all.
3. As seen from above table, the value of z jumps from 0 to 6 to – 2 to 8, *i.e.*, there are up's and down's.

These disadvantages are overcome by *simplex method* yielding successive solutions with progressively improving the value of z, culminating into the optimal one.

Q. 1. Explain the procedure of generating extreme point solutions to a linear programming problem pointing out the assumptions made, if any.

2. Compute all the basic feasible solutions of the LP problem : Max $z = 2x_1 + 3x_2 + 4x_3 - 7x_4$ s.t. $2x_1 + 3x_2 - x_3 + 4x_4 = 8$
$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$
and choose that one which maximizes z.

[IAS (Main) 2001 (Type); Rewa (MP) 93 ; Roorkee (B.E. IVth) 91 ; Meerut (B.Sc.) 90]

SELF-EXAMINATION PROBLEMS

1. Prove that a vertex is a boundary point but all boundary points are not vertices. Give examples. Identify the vertices, if any, of the following sets :
(a) $\{\mathbf{x} : |\mathbf{x}| \le 1, \mathbf{x} \in R^n\}$ (b) $\{\mathbf{x} : \mathbf{x} = (1 - \lambda)\mathbf{x}_1 + \lambda \mathbf{x}_2, \lambda \ge 0, \mathbf{x}_1, \mathbf{x}_2 \in R^n\}$.
[**Ans.** (a) Vertex is 1, (b) The point **x** will be vertex when there does not exist any pair of points $\mathbf{x}_1$, $\mathbf{x}_2$ which satisfies the given conditions for $0 < \lambda < 1$]

2. Which of the following sets are convex; if so, why ?
(i) $X = \{(x_1, x_2) ; x_1 x_2 \le 1, x_1 \ge 0, x_2 \ge 0\}$
(ii) $X = \{(x_1, x_2) : x_2^2 - 3 \ge -x_1^2 : x_1 \ge 0$ and $x_2 \ge 0\}$ **[Delhi B.Sc. 93]**
(iii) $X = \{(x_1, x_2) : x_1^2 + x_2^2 \le 9\}$,
(iv) $X = \{(x_1, x_2) : x_2^2 \le 4x_1 ; x_1, x_2 \ge 0\}$ **[Delhi B.Sc. 90]**
[**Ans.** (i) Concave (ii) Not Convex (iii) Convex (iv) Convex]

3. Discuss whether the following sets are convex or not, and find the convex hull of the set in each case :
(i) Set of points on the line $y = mx + c$,
(ii) Set of points of the union of the half lines $x = 0$, $y > 0$, $y = 0$ $x \ge 0$ on the xy plane.
(iii) Points (0, 0), (0, 1), (1, 0), (1, 1) on the (x, y) plane.
[**Ans.** (i) Not convex (ii) convex (iii) convex].

4. Prove that the set $B = \{(x_1, x_2) : x_1^2 + x_2^2 \le 4\}$ is a convex set.

5. Define a convex set. Show that the set $S = \{(x_1, x_2) : 3x_1^2 + 2x_2^2 \le 6\}$ is convex.

6. Determine the convex hull of the following sets : (i) $X = \{(x_1, x_2) + x_1^2 : x_2^2 = 1\}$ (ii) $X = \{(x_1, x_2)\}$.
[**Ans.** (i) $\langle X \rangle = \{(x_1, x_2) : x_1^2 + x_2^2 \le 1\}$ (ii) $\langle X \rangle = \{\mathbf{x} : \mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 ; 0 \le \lambda \le 1\}$

7. Graph the convex hull of the points : (0, 0), (0, 1), (1, 2), (1, 1), (4, 0).
Which of these points in an interior point of the convex hull ? Express it as a convex combination of the extreme points.
[**Ans.** (1, 1) is the interior point of convex hull].

8. Sketch the convex polygon spanned by the following points in a two-dimensional Euclidean space. Which of these points are vertices ? Express the others the convex linear combination of the vertices.
(0, 0), (0, 1), (1, 0), (1/2, 1/4).
[**Ans.** $\frac{1}{4}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_2 + \frac{1}{2}\mathbf{x}_3$ where $\mathbf{x}_1 = (0, 0)$, $\mathbf{x}_2 = (0, 1)$ and $\mathbf{x}_3 = (1, 0)$, which are vertices].

9. Prove that the following sets are convex. Which are convex polyhedron sets ?
(i) The interrior plus the edge of a triangle. (ii) The interior of a circle. (iii) The interior of a rectangle. (iv) A rectangle surmounted by a semicircle.

10. Consider the plane with a cartesian coordinate system. A rectangle with sides a_1 and a_2 ($a_1 \ne a_2$) is placed with one corner at the origin and two of its sides along the axes. Prove that the interior of the rectangle plus its edges form a polyhedron convex set.

11. Find the extreme points of the convex polygon given by the inequalities :
$$2x_1 + x_2 + 9 \ge 0, -x_1 + 3x_2 + 6 \ge 0, x_1 + 2x_2 - 3 \le 0, x_1 + x_2 \le 9.$$

12. Prove that the objective function of a linear programming problem assumes its minimum value at an extreme point of the convex set X generated by the set of all feasible solutions.
[**Hint.** Proceed as in Theorem 2.8, page 2.54].

13. Prove that the extreme point of the feasible region of a linear programming problem correspond to the feasible solutions of the problem.

14. If a linear programming problem, Max **AX = b**, **X ≥ 0**, has at least an optimal feasible solution, then at least one basic feasible solution must be optimal. **[Meerut (B.Sc.) 90]**

15. When is a set K $\subset E^n$ said to be convex ? Show that for a set K to be convex it is necessary and sufficient that every convex linear combination of points in K belong to K. What is the role of the theory of convex sets in the solution of linear programming problems ? **[I.A.S. (Maths) 90]**

SELF-EXAMINATION QUESTIONS

1. Write a short note on extreme point solutions of L.P.P.
2. What do you mean by a convex combination of a finite number of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$? Sketch on the graph paper, the convex polyhedron generated by the following sets of points :
   (i) (3, 4), (5, 6), (0, 0), (2, 2), (1, 0), (2, 5), (4, 7). (ii) (– 1, 2), (3, – 4), (4, 4), (0, 0), (6, 5), (7, 1).
3. Define a convex set. Prove that the set of all feasible solutions of a L.P.P. is a convex set.
4. Indicate whether the following statements are true. If not, write their correct forms :
   (i) A line passing through two distinct points $\mathbf{x}_1$ and $\mathbf{x}_2$ is the set of all those points $\mathbf{x}$ such that $\mathbf{x}_2 = \lambda\mathbf{x} + (1 - \lambda)\, \mathbf{x}_1$ where $\lambda \in [0, 1]$.
   (ii) The number of edges that can emanate from any given extreme point of convex set of feasible solutions is two.
   (iii) The number of extreme points for a L.P. problem : Max. $\mathbf{z} = \mathbf{CX}$ subject to $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, where $\mathbf{A}$ is an $m \times n$, is equal to $n!/m!$ ·
5. Define a convex function f on a convex set S in R^n. Show that the points of S on which f takes on its global minimum is a convex set. **[Agra M.Sc. (Math) 98]**
6. Compute all basic fecible solutions of the linear programming problem :
   Max. $z = 2x_1 + 3x_2 + 2x_3$, subject to $2x_1 + 3x_2 - x_3 = 8$, $x_1 - 2x_2 + 6x_3 = -3$, $x_1, x_2, x_3 \geq 0$ and hence indicate the optimum solution. **[IAS (Main) 2001]**

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5

LINEAR PROGRAMMING PROBLEM : SIMPLEX METHOD

5·1. INTRODUCTION

It has not been possible to obtain the graphical solution to the LP problem of more than two variables. The analytic solution is also not possible because the tools of analysis are not well suited to handle inequalities. In such cases, a simple and most widely used simplex method is adopted which was developed by ***G. Dantzig*** in 1947.

The *simplex method*† provides an ***algorithm*** (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the ***fundamental theorem of linear programming.***

It is clear from Fig. 3.4 (page 78) that feasible solutions may be *infinite* in number (because there are infinite number of points in the feasible region, *OABCD*). So, it is rather impossible to search for the optimum solution amongst all the feasible solutions. But fortunately, the number of basic feasible solutions are finite in number (which are corresponding to extreme points O, A, B, C, D, respectively). Even then, a great labour is required in finding all the basic feasible solutions and to select that one which optimizes the objective function.

The simplex method provides a systematic algorithm which consists of moving from one basic feasible solution (one vertex) to another in a prescribed manner so that the value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated. If the objective function is improved at each jump, then no basis can ever repeat and there is no need to go back to vertex already covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps. The procedure is explained in detail through a numerical example (*see Example* 2, ch. 5, page 70 (Unit 2)).

The simplex algorithm is an iterative (step-by-step) procedure for solving LP problems. It consists of—

(i) having a trial basic feasible solution to constraint-equations,

(ii) testing whether it is an optimal solution,

(iii) improving the first trial solution by a set of rules, and repeating the process till an optimal solution is obtained.

The computational procedure requires at most m [equal to the number of equations in (3·12)] non-zero variables in the solution at any step. In case of less than m non-zero variables at any stage of computations the degeneracy arises in LP problem. The case of degeneracy has also been discussed in detail in *this chapter.*

Further, it is very interesting to note that a feasible solution at any iteration is related to the feasible solution at the successive iteration in the following way. One of the non-basic variables (which are zero now) at one iteration becomes *basic* (non-zero) at the following iteration, and is called an ***entering variable.*** To compensate, one of the basic variables (which are non-zero now) at one iteration becomes non-basic (zero) at the following iteration, and is called a ***departing variable.*** The other non-basic variables remain zero, and the other basic variables, in general, remain non-zero (though their values may change).

For convenience, re-state the LP problem in standard form :

$$\text{Max. } z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \dots + 0x_{n+m} \quad \text{...(5.1)}$$

subject to the constraints :

$$\left.\begin{array}{llllll} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & + x_{n+1} & & & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & & + x_{n+2} & & & = b_2 \\ \dots \quad \dots & \dots & \dots & \dots & & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & & & & + x_{n+m} & = b_m \end{array}\right\} \quad \text{...(5.2)}$$

† For complete development of **'Simplex Method'** please see Appendix-A (**Theory of Simplex Method**) on page **1119.**

and $$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, x_{n+1} \geq 0, \ldots, x_{n+m} \geq 0 \qquad \ldots(5.3)$$

For easiness, an obvious starting basic feasible solution of m equations (5·2) is usually taken as : $x_1 = x_2 = x_3 = \ldots = x_n = 0$; $x_{n+1} = b_1, x_{n+2} = b_2, \ldots, x_{n+m} = b_m$. For this solution, the value of the objective function (5·1) is zero. Here $x_1, x_2, x_3, \ldots, x_n$ (each equal to zero) are ***non-basic variables*** and remaining variables $(x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+m})$ are ***basic variables*** (some of them may also have the value zero).

5.2. SOME MORE DEFINITIONS AND NOTATIONS

The first basic feasible solution is : $x_1 = x_2 = x_3 = \ldots = x_n = 0$; and $x_{n+1} = b_1$, $x_{n+2} = b_2$, $x_{n+3} = b_3 \ldots$, $x_{n+m} = b_m$ for the reformulated LP problem : Max $\mathbf{z} = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$ and $\mathbf{X} \geq \mathbf{0}$.

First denote the *j*th column of $m \times (n+m)$ matrix A by $\mathbf{a}_j$ $(j = 1, 2, 3, \ldots, n+m)$, so that

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m+n}]. \qquad \ldots(5.4)$$

Now form an $m \times m$ non-singular matrix B, called ***basis matrix***, whose column vectors are m linearly independent columns selected from matrix A and renamed as $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$. Therefore,

$$\mathbf{B} = [\beta_1, \beta_2, \ldots, \beta_m] = [\mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \ldots, \mathbf{a}_{n+m}] \qquad \ldots(5.5)$$

For initial basic feasible solution,

$$\mathbf{B} = [(1, 0, 0, \ldots, 0), (0, 1, 0, 0 \ldots, 0), \ldots, (0, 0, \ldots, 1)] = I_m \text{ (identity matrix)}.$$

The matrix **B** is evidently a basis matrix because column vectors in **B** form a basis set of m-dimensional Euclidean space (E^m).

Second, denote the basic variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ by $x_{B1}, x_{B2}, \ldots, x_{Bm}$ respectively, to give the basic feasible solution in the form :

$$\mathbf{X_B} = (x_{B1}, x_{B2}, x_{B3}, \ldots, x_{Bm}) = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+m}) \qquad \ldots(5.6)$$

For initial basic feasible solution,

$$\mathbf{X}_B = (b_1, b_2, b_3, \ldots, b_m) = \text{ right side constants of (5.2).}$$

Next, the coefficients of basic variables $x_{B1}, x_{B2}, \ldots, x_{Bm}$ in the objective function z will be denoted by $c_{B1}, c_{B2}, \ldots, c_{Bm}$ respectively, so that

$$\mathbf{C}_B = (c_{B1}, c_{B2}, \ldots, c_{Bm}).$$

For initial basic feasible solution,

$$\mathbf{C}_B = (0, 0, \ldots, 0) = \mathbf{0} \text{ (null vector)}$$

Consequently, the objective function

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \ldots + 0x_{n+m} \text{ becomes}$$

$$z = c_1.0 + c_2.0 + \ldots + c_n.0 + c_{B1}x_{B1} + \ldots + c_{Bm}x_{Bm} \qquad [\text{since } x_1 = x_2 = x_3 = \ldots = x_n = 0]$$

or $$z = \mathbf{C}_B \mathbf{X}_B. \qquad \ldots(5.7)$$

Because $\mathbf{C}_B = \mathbf{0}$ (null vector) for initial solution, therefore

$$z = 0, \mathbf{X_B} = \mathbf{b}.$$

Since **B** is an $m \times m$ non-singular basis matrix, any vector in E^m can be expressed as a linear combination of vectors in **B** (by definition of basis for vector space). In particular, each vector $\mathbf{a}_j$ $(j = 1, 2, \ldots, n+m)$ of matrix **A** can be expressed as a linear combination of vectors β_i $(i = 1, 2, \ldots, m)$ in **B**. The notation for such linear combination is given by

$$\mathbf{a}_j = x_{1j}\beta_1 + x_{2j}\beta_2 + \ldots + x_{mj}\beta_m = (\beta_1, \beta_2, \ldots, \beta_m)\begin{bmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{bmatrix} = \mathbf{BX_j} \qquad \ldots(5.8)$$

where x_{ij} $(i = 1, 2, 3, \ldots, m)$ are scalars required to express each $\mathbf{a}_j$ $(j = 1, 2, 3, \ldots, n+m)$ as linear combination of basis vectors $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$.

Therefore, $\mathbf{X_j} = \mathbf{B}^{-1}\mathbf{a_j}$ and hence matrix $(\mathbf{X_j})$ will change if the columns of (A) forming (B) change.

For initial solution, $\mathbf{a}_j = \mathbf{I}_m\mathbf{X}_j = \mathbf{X}_j$.

Next define a new variable, say z_j, as

$$z_j = x_{1j}\, c_{B1} + x_{2j}\, c_{B2} + \ldots + x_{mj}c_{Bm} = \sum_{i=1}^{m} c_{Bi}\, x_{ij} = \mathbf{C}_B\, \mathbf{X}_j. \qquad \ldots(5.9)$$

Δ_j denotes the *net evaluation* which is computed by the formula :

$$\Delta_j = z_j - c_j = \mathbf{C_B X}_j - c_j \quad \text{...(5.10)}$$

Lastly, these notations can be summerized in the following Starting Simplex Table 5.1.

Table 5.1 : Starting Simplex Table

		$c_j \rightarrow$	c_1	c_2	...	c_n	0	0	...	0	
BASIC VARIABLES	$\mathbf{C}_B$	$\mathbf{X}_B$	$\mathbf{X}_1 (= \mathbf{a}_1)$	$\mathbf{X}_2 (= \mathbf{a}_2)$	...	$\mathbf{X}_n (= \mathbf{a}_n)$	$\mathbf{X}_{n+1}$ (β_1)	$\mathbf{X}_{n+2}$ (β_2)	...	$\mathbf{X}_{n+m}$ (β_m)	MIN RATIO
$x_{n+1} (= s_1)$	$c_{B1} (= 0)$	$x_{B1} (= b_1)$	$x_{11} (= a_{11})$	$x_{12} (= a_{12})$	...	$x_{1n} (= a_{1n})$	1	0	...	0	
$x_{n+2} (= s_2)$	$c_{B2} (= 0)$	$x_{B2} (= b_2)$	$x_{21} (= a_{21})$	$x_{22} (= a_{22})$	...	$x_{2n} (= a_{2n})$	0	1	...	0	
:	:	:	:	:	:	:	:	:	:	:	
$x_{n+m} (= s_m)$	$c_{Bm} (= 0)$	$x_{Bm} (= b_m)$	$x_{m1} (= a_{m1})$	$x_{m2} (= a_{m2})$	...	$x_{mn} (= a_{mn})$	0	0	...	1	
	$z = \mathbf{C}_B\mathbf{X}_B$		Δ_1	Δ_2	...	Δ_n	0	0	...	0	$\Delta_j = \mathbf{C}_B\mathbf{X}_j - c_j$

Note. Basic variables in the first column are always sequenced in the order of columns forming the unit matrix.

Above definitions and notations can be clearly understood by the following numerical example.

5.2-1. An Example to Explain Above Definitions and Notations

Example 1. *Illustrate definitions and notations by the linear programming problem :*

Maximize $z = x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5$, *subject to* $4x_1 + 2x_2 + x_3 + x_4 = 4$, $x_1 + 2x_2 + 3x_3 - x_5 = 8$.

Solution. First of all, constraint equations in matrix form may be written as

$$\overset{\mathbf{A}}{\begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ \end{matrix}} \begin{bmatrix} 4 & 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & -1 \end{bmatrix} \overset{\mathbf{X}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \overset{\mathbf{B}}{\begin{bmatrix} 4 \\ 8 \end{bmatrix}}$$

or $$\mathbf{AX} = \mathbf{b}.$$

A basis matrix $\mathbf{B} = (\beta_1, \beta_2)$ is formed using columns $\mathbf{a}_3$ and $\mathbf{a}_1$, so that

$$\beta_1 = \mathbf{a}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \beta_2 = \mathbf{a}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The rank of matrix $\mathbf{A}$ is 2, and hence $\mathbf{a}_3$, $\mathbf{a}_1$ column vectors are linearly independent, and thus forms a basis for R^2.

Thus, basis matrix is $$\mathbf{B} = (\beta_1, \beta_2) = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} \quad (\text{columns } \mathbf{a}_3, \mathbf{a}_1)$$

Using (5·4) and (5.8), the basic feasible solution is

$$\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b} = \left[\frac{1}{|\mathbf{B}|} \text{adj } (\mathbf{B})\right] \mathbf{b} = \frac{-1}{11} \begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 28 \\ 4 \end{bmatrix}$$

or $$\mathbf{X_B} = \begin{bmatrix} 28/11 \\ 4/11 \end{bmatrix} = \begin{bmatrix} x_{B1} \\ x_{B2} \end{bmatrix}.$$

Therefore, basic variables are $x_{B1} = 28/11 = x_3$, $x_{B2} = 4/11 = x_1$, and remaining variables are non-basic (which are always zero) *i.e.*, $x_2 = x_4 = x_5 = 0$. Also,

$$c_{B1} = \text{coefficient of } x_{B1} = \text{coeff. of } x_3 = c_3 = 3$$

c_{B2} = coefficient of x_{B2} = coeff. of $x_1 = c_1 = 1$

Hence $\mathbf{c_B} = (3, 1)$.

Now, using (5.7), the value of the objective function is

$$z = \mathbf{c_B X_B} = (3, 1)\begin{pmatrix} 28/11 \\ 4/11 \end{pmatrix} = \frac{88}{11}.$$

Also, any vector $\mathbf{a_j} = (j = 1, 2, 3, 4, 5)$ can be expressed as linear combination of vectors β_i $(i = 1, 2)$. Therefore, to express $\mathbf{a}_2$ as linear combination of β_1, β_2, we have

$$\mathbf{a}_2 = x_{12}\beta_1 + x_{22}\beta_2 = x_{12}\mathbf{a}_3 + x_{22}\mathbf{a}_1.$$

To compute values of scalars x_{12} and x_{22}, use the result (5.3) to get

$$\mathbf{X}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = -\frac{1}{11}\begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Therefore $x_{12} = 6/11, x_{22} = 4/11$.

Similar treatment can be adopted for expressing other $\mathbf{a}_j$'s as linear combinations of β_1 and β_2.

Now, using (5.6b), the variable z_2 corresponding to vector $\mathbf{a}_2$ can be obtained as

$$z_2 = \mathbf{c_B X_2} = (3, 1)\begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \left(3 \times \frac{6}{11} + 1 \times \frac{4}{11}\right) = \frac{22}{11}.$$

Similarly z_1, z_3, z_4, z_5 can also be computed.

5·3. COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

The computational aspect of the simplex procedure is first explained by the following simple example.

Example 2. *Consider the linear programming problem :*

Maximize $z = 3x_1 + 2x_2$, subject to the constraints :

$x_1 + x_2 \le 4, x_1 - x_2 \le 2,$ *and* $x_1, x_2 \ge 0$. **[Kanpur 2000, 96; IAS (Maths.) 92]**

Solution. ***Step 1.*** First, observe whether all the right side constants of the constraints are non-negative. If not, it can be changed into positive value on multiplying both sides of the constraints by -1. In this example, all the b_i's (right side constants) are already positive.

Step 2. Next convert the inequality constraints to equations by introducing the non-negative ***slack*** or ***surplus*** variables. The coefficients of slack or surplus variables are always taken zero in the objective function. In this example, all inequality constraints being '≤', only slack variables s_1 and s_2 are needed. Therefore, given problem now becomes :

Maximize $z = 3x_1 + 2x_2 + 0s_1 + 0s_2$, subject to the constraints :

$$x_1 + x_2 + s_1 \quad = 4$$
$$x_1 - x_2 \quad + s_2 = 2$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Step 3. Now, present the constraint equations in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Step 4. Construct the starting simplex table using the notations already explained in ***Sec 5·2.***

It should be remembered that the values of non-basic variables are always zero at each iteration. So $x_1 = x_2 = 0$ here. Column $\mathbf{X_B}$ gives the values of basic variables as indicated in the first column. So $s_1 = 4$ and $s_2 = 2$ here. The complete starting basic feasible solution can be immediately read from Table 5·2 as : $s_1 = 4, s_2 = 2, x_1 = 0, x_2 = 0$, and the value of the objective function is *zero.*

Note. In this step, the variables s_1 and s_2 are corresponding to the columns of basis matrix (identity matrix), so will be called ***basic variables.*** Other variables, x_1 and x_2, are *non-basic variables* which always have the value zero.

Table 5·2 : Starting Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	$X_3 (S_1)$ (β_1)	$X_4 (S_2)$ (β_2)	MIN. RATIO X_B/X_k for $X_K > 0$
		$c_j \rightarrow$	3	2	0	0	
					BASIS MATRIX		
s_1	0	4	1	1	1	0	TO BE COMPUTED IN NEXT STEP.
s_2	0	2	1	−1	0	1	
	$z = C_B X_B$		$\Delta_1 = -3$ ↑	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = z_j - c_j = C_B X_j - c_j$

Step 5. Now, proceed to test the basic feasible solution for optimality by the rules given below. This is done by computing the *'net evaluation'* Δ_j for each variable x_j (column vector X_j) by the formula

$$\Delta_j = z_j - c_j = C_B X_j - c_j \qquad \text{[from (5·10)]}$$

Thus, we get

$$\Delta_1 = C_B X_1 - c_1 = (0, 0)(1, 1) - 3 = (0 \times 1 + 0 \times 1) - 3 = -3$$

$$\Delta_2 = C_B X_2 - c_2 = (0, 0)(1, -1) - 2 = (0 \times 1 - 0 \times 1) - 2 = -2$$

$$\Delta_3 = C_B X_3 - c_3 = (0, 0)(1, 0) - 0 = (0 \times 1 + 0 \times 0) - 0 = 0$$

$$\Delta_4 = 0$$

Remark. Note that in the starting simplex table Δ_j's are same as $(-c_j)$'s. Also, Δ_j's corresponding to the columns of unit matrix (basis matrix) are always zero. So there is no need to calculate them.

Optimality Test :

(i) If all $\Delta_j (= z_j - c_j) \geq 0$, the solution under test will be ***optimal. Alternative optimal solutions*** will exist if any non-basic Δ_j is also zero.

(ii) If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.

(iii) If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero (≤ 0), then the solution under test will be ***unbounded.***

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that Δ_1 and Δ_2 both are negative. Hence, we have to proceed to improve this solution in ***Step 6.***

Step 6. In order to improve this basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as ***'incoming vector'*** and ***'outgoing vector'*** respectively.

'Incoming vector'. The incoming vector X_k is always selected corresponding to the most negative value of Δ_j (say, Δ_k). Here $\Delta_k = \min [\Delta_1, \Delta_2] = \min [-3, -2] = -3 = \Delta_1$. Therefore, $k = 1$ and hence column vector X_1 must enter the basis matrix. The column X_1 is marked by an upward arrow (↑).

'Outgoing vector'. The outgoing vector β_r is selected corresponding to the minimum ratio of elements of X_B by the corresponding *positive* elements of predetermined incoming vector X_k. This rule is called the ***Minimum Ratio Rule.*** In mathematical form, this rule can be written as

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Br}}{x_{ik}}, x_{ik} > 0\right]$$

For $k = 1$,

$$\frac{x_{Br}}{x_{r1}} = \min\left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{4}{1}, \frac{2}{1}\right]$$

or

$$\frac{x_{Br}}{x_{r1}} = \frac{2}{1} = \frac{x_{B2}}{x_{21}}.$$

Comparing both sides of this equation, we get $r = 2$. So the vector β_2, *i.e.*, X_4 marked with downward arrow (↓) should be removed from the basis matrix. The ***Starting Table 5·2*** is now modified to ***Table 5·3*** given below.

Table 5·3

key element

BASIC VARIABLES	Ca	Xa	X_1 (3)	X_2 (2)	$X_3(S_1)$ (β_1) (0)	$X_4(S_2)$ (β_2) (0)	MIN. RATIO (X_B/X_1)
s_1	0	4	1	1	1	0	4/1
s_2	0	2	[1]	1	0	1	2/1 ← MIN. RATIO
	$z = C_B X_B = 0$		−3 (min. Δ_j)	−2	0	0	← $\Delta_j = z_j - c_j = C_B B_j - c_j$

↑ entering vector ↓ leaving vector

Step 7. In order to bring $\beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in place of incoming vector $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, unity must occupy in the marked '☐' position and zero at all other places of X_1. If the number in the marked '☐' position is other than *unity, divide all elements of that row by the 'key element'. (The element at the intersection of minimum ratio arrow (←) and incoming vector arrow (↑) is called the **key element or pivot element**).*

Then, subtract appropriate multiplies of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column X_1. Thus, the process can be fortified by simple matrix transformation as follows :

The intermediate coefficient matrix is :

	X_B	X_1	X_2	X_3	X_4	
R_1	4	1	1	1	0	
R_2	2	1	−1	0	1	
R_3	$z = 0$	−3	−2	0	0	← Δ_j

Apply $R_1 \to R_1 - R_2$, $R_3 \to R_3 + 3R_2$ to obtain

X_B	X_1	X_2	X_3	X_4	
2	0	2	1	-1	
2	1	−1	0	1	
$z = 6$	0	−5	0	3	← Δ_j

Now, construct the improved simplex table as follows :

Table 5·4

BASIC VARIABLES	C_B	X_B	X_1 (β_2) (3)	X_2 (2)	$X_3(S_1)$ (β_1) (0)	$X_4(S_2)$ (0)	MIN-RATIO (X_B/X_2, $X_2 > 0$)
s_1	0	2	0	[2]	1	1	$\frac{2}{2}$ ← key row
x_1	3	2	1	−1	0	1	~~2/−1~~ (negative ratio is not counted)
	$z = C_B X_B = 6$		0	−5 ↑	0 ↓	3	← Δ_j

key column

From this table, the improved basic feasible solution is read as : $x_1 = 2$, $x_2 = 0$, $s_1 = 2$, $s_2 = 0$. The improved value of $z = 6$.

It is of particular interest to note here that Δ_j's are also computed while transforming the table by matrix method. However, the correctness of Δ_j's can be verified by computing them independently by using the formula $\Delta_j = C_B X_j - c_j$.

Step 8. Now repeat *Steps* 5 *through* 7 as and when needed until an optimum solution is obtained in Table 5·5.

$$\Delta_k = \text{most negative } \Delta_j = -5 = \Delta_2 .$$

Therefore, $k = 2$ and hence $\mathbf{X_2}$ should be the entering vector (key column). By minimum ratio rule :

$$\text{Minimum Ratio}\left(\frac{\mathbf{X_B}}{\mathbf{X_2}}, \mathbf{X_2} > \mathbf{0}\right) = \text{Min}\left[\frac{2}{2}, -\right]$$ (since negative ratio is not counted, so the second ratio is not considered)

Since *first ratio* is minimum, remove the first vector β_1 from the basis matrix. Hence the key element is 2. Dividing the first row by key element 2, the intermediate coefficient matrix is obtained as :

	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	
R_1	1	0	[1]	1/2	− 1/2	
R_2	2	1	− 1	0	1	
R_3	$z = 6$	0	− 5	0	3	← Δ_j

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 5R_1$

1	0	1	1/2	− 1/2	
3	1	0	1/2	1/2	
$z = 11$	0	0	5/2	1/2	← Δ_j

Now construct the next improved simplex table as follows :

Final Simplex Table 5·5

		$c_j \to$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$ (β_2)	$\mathbf{X_2}$ (β_1)	$\mathbf{S_1}$	$\mathbf{S_2}$	
$\to x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
	$z = \mathbf{C_B}\, \mathbf{X_B} = \mathbf{11}$		0	0	5/2	1/2	← Δ_j

The solution as read from this table is : $x_1 = 3$, $x_2 = 1$, $s_1 = 0$, $s_2 = 0$, and max. $z = 11$. Also, using the formula $\Delta_j = \mathbf{C_B}\, \mathbf{X_j} - c_j$ verify that all Δ_j's are non-negative. Hence the optimum solution is

$$x_1 = 3\ , x_1 = 1\ ; \max \mathbf{z} = 11\ .$$

Note. If at the optimal stage, it is desired to bring s_1 in the solution, the total profit will be reduced from 11 (the optimal value) to 5/2 times of 2 units of s_1 in Table 3·4, i.e., $z = 11 - 5/2 \times 2 = 6$. This explains the *economic interpretation* of net-evaluations Δ_j .

5·4. SIMPLE WAY FOR SIMPLEX METHOD COMPUTATIONS

Complete solution with its different computational steps can be more conveniently represented by the following single table (see Table 5.6).

Table 5·6

		$c_j \to$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	MIN RATIO ($\mathbf{X_B}/\mathbf{X_K}$)
s_1	0	4	1 ↑	1	1	0	4/1
← s_2	0	2	← [1]	1	0	1	2/1 ← Min
$x_1 = x_2 = 0$	$z = \mathbf{C_B X_B} = 0$		− 3* ↑	− 2	0	0 ↓	← $\Delta_j = z_j - c_j$
← s_1	0	2	0	[2]	1	−1	2/2 Min ←
→ x_1	3	2	1	−1	0	1	—
$x_2 = s_2 = 0$	$z = \mathbf{C_B X_B} = 6$		0	−5* ↑	0 ↓	3	← Δ_j
→ x_2	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
$s_1 = s_2 = 0$	$z = \mathbf{C_B X_B} = \mathbf{11}$		0	0	5/2	1/2	← All $\Delta_j \geq 0$

Thus, the optimal solution is obtained as : $x_1 = 3$, $x_2 = 1$, max $z = 11$.

Q. 1. What is a simplex ? Describe simplex method of solving linear programming problems.
2. Write the steps used in the simplex method. **[Kanpur (B. Sc.) 90]**
3. Describe a computational procedure of the simplex method for the solution of a maximization l.p.p.

Tips for Quick Solution :

1. In the first iteration only, since Δ_j's are the same as $-c_j$'s, so there is no need of calculating them separately by using the formula $\Delta_j = \mathbf{C_B X_j} - c_j$.
2. Mark *min* (Δ_j) by '↑' which at once indicates the column $\mathbf{X}_k$ needed for computing the minimum ratio $(\mathbf{X_B}/\mathbf{X}_k)$.
3. '*Key element*' is found at the place where the upward directed arrow '↑' of min Δ_j and the left directed arrow (←) of minimum ratio $(\mathbf{X_B}/\mathbf{X}_k)$ intersect each other in the simplex table.
4. '*Key element*' indicates that the current table must be transformed in such a way that the key element becomes 1 and all other elements in that column become 0.
5. Since Δ_j's corresponding to unit column vectors are always zero, there is no need of calculating them.
6. While transforming the table by row operations, the value of z and corresponding Δ_j's are also computed at the same time. Thus a lot of time and labour can be saved in adopting this technique.

Example 3. *Min* $z = x_1 - 3x_2 + 2x_3$, *subject to* :

$$3x_1 - x_2 + 3x_3 \le 7, -2x_1 + 4x_2 \le 12, -4x_1 + 3x_2 + 8x_3 \le 10, \textit{ and } x_1, x_2, x_3 \ge 0.$$

[Kanpur (B.Sc.) 95, 93, (B.A.) 90]

Solution. This is the problem of minimization. Converting the objective function from minimization to maximization, we have

$$\text{Max.} -z = -x_1 + 3x_2 - 2x_3 = \text{Max. } z' \text{ where } -z = z',$$

Here we give only tables of solution. The students are advised to verify them.

Table 5·7. Simplex Table

BASIC VARIABLES	$\mathbf{C_B}$	$c_j \rightarrow$ $\mathbf{X_B}$	-1 $\mathbf{X_1}$	3 $\mathbf{X_2}$	-2 $\mathbf{X_3}$	0 $\mathbf{X_4}$	0 $\mathbf{X_5}$	0 $\mathbf{X_6}$	MIN. RATIO $(\mathbf{X_B}/\mathbf{X}_k)$
x_4	0	7	3	-1	3	1	0	0	—
$\leftarrow x_5$	0	12	-2	[4]	0	0	1	0	$12/4 \leftarrow$ min.
x_6	0	10	-4	3	8	0	0	1	10/3
$x_1 = x_2 = x_3 = 0$	$z' = 0, z = 0$		1	-3^* ↑	2	0 ↓	0	0	$\leftarrow \Delta_j$
$\leftarrow x_4$	0	10	[5/2]	0	3	1	1/4	0	$\frac{10}{5/2} \leftarrow$
$\rightarrow x_2$	3	3	$-1/2$	1	0	0	1/4	0	—
x_6	0	1	$-5/2$	0	8	0	$-3/4$	1	—
$x_1 = x_3 = x_5 = 0$	$z' = 9$ $\therefore z = -9$		$-1/2^*$ ↑	0	2	0 ↓	3/4	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	-1	4	1	0	6/5	2/5	1/10	0	
x_2	3	5	0	1	3/5	1/5	3/10	0	
x_6	0	11	0	0	11	1	$-1/2$	1	
$x_3 = x_4 = x_5 = 0$	$z' = 11$ $\therefore z = -11$		0	0	13/5	1/5	8/10	0	$\leftarrow \Delta_j \ge 0$

The optimal solution is : $x_1 = 4, x_2 = 5, x_3 = 0$, *Min* $z = -11$.

Example 4. *Max.* $z = 3x_1 + 2x_2 + 5x_3$, *subject to the constraints :*

$$x_1 + 2x_2 + x_3 \le 430, 3x_1 + 2x_3 \le 460, x_1 + 4x_2 \le 420, \textit{ and } x_1, x_2, x_3 \ge 0.$$

[IAS (Main 94)]

Solution.

Table 5·8. Simplex Table

BASIC VARIABLES	C_B	X_B	X_1 (3)	X_2 (2)	X_3 (5)	X_4 (0)	X_5 (0)	X_6 (0)	MIN RATIO (X_B/X_k)
x_4	0	430	1	2	1	1	0	0	430/1
$\leftarrow x_5$	0	460	3	0	[2]	0	1	0	460/2 ←
x_6	0	420	1	4	0	0	0	1	—
$x_1 = x_2 = x_3 = 0$		$z = 0$	–3	–2	–5* ↑	0	0 ↓	0	$\leftarrow \Delta_j$
$\leftarrow x_4$	0	200	– 1/2	[2]	0	1	–1/2	0	200/2 ←
$\rightarrow x_3$	5	230	3/2	0	1	0	1/2	0	—
x_6	0	420	1	4	0	0	0	1	420/4
$x_1 = x_2 = x_5 = 0$		$z = 1150$	9/2	–2* ↑	0	0 ↓	5/2	0	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	100	– 1/4	1	0	1/2	– 1/4	0	
x_3	5	230	3/2	0	1	0	1/2	0	
x_6	0	20	2	0	0	–2	1	1	
$x_1 = x_4 = x_5 = 0$		$z = 1350$	4	0	0	1	2	0	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, the solution is : $x_1 = 0$, $x_2 = 100$, $x_3 = 230$, max $z = 1350$.

Example 5. *Solve the LP problem :Max.* $z = 3x_1 + 5x_2 + 4x_3$, *subject to the constraints :*

$$2x_1 + 3x_2 \leq 8,\ 2x_2 + 5x_3 \leq 10,\ 3x_1 + 2x_2 + 4x_3 \leq 15,\ \textit{and}\ x_1, x_2, x_3 \geq 0 .$$

[Tamilnadu (ERODE) 97; Rewa 93; Kanpur (B.Sc.) 92; (B.A.) 90, Meerut (M.Sc. Stat. & B.Sc. Math.) 90]

Solution. After introducing slack variables, the constraint equations become :

$$\begin{aligned} 2x_1 + 3x_2 \quad + x_4 \quad &= 8 \\ 2x_2 + 5x_3 \quad + x_5 \quad &= 10 \\ 3x_1 + 2x_2 + 4x_3 \quad + x_6 &= 15. \end{aligned}$$

Table 5·9. Starting Simplex Table

BASIC VARIABLES	C_B	X_B	X_1 (3)	X_2 (5)	X_3 (4)	X_4 (0)	X_5 (0)	X_6 (0)	MIN RATIO. (X_B/X_2)
$\leftarrow x_4$	0	8	2	[3]	0	1	0	0	8/3 ←
x_5	0	10	0	2	5	0	1	0	10/2
x_6	0	15	3	2	4	0	0	1	15/2
$x_1 = x_2 = x_3 = 0$		$z = C_B X_B = 0$	–3	–5* ↑ Incoming vector	–4	0 ↓ outgoing vector	0	0	$\leftarrow \Delta_j$

Now apply short-cut method for minimum ratio rule (min X_B/X_2), and find the key element 3. This key element indicates that unity should be at first place of X_2 , so the vector to be removed from the basis matrix is X_4.

Now, in order to get the second simplex table, calculate the intermediate coefficient matrices as follows :

First, divide the first row by 3 to get

R_1	8/3	2/3	1	0	1/3	0	0	
R_2	10	0	2	[5]	0	1	0	
R_3	15	3	2	4	0	0	1	
R_4	0	– 3	– 5	– 4	0	0	0	$\leftarrow \Delta_j$

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 2R_1$, $R_4 \rightarrow R_4 + 5R_1$,

R_1	8/3	2/3	[1]	0	1/3	0	0	
R_2	14/3	– 4/3	0	5	– 2/3	1	0	
R_3	29/3	5/3	0	4	– 2/3	0	1	
R_4	40/3	1/3	0	-4	5/3	0	0	$\leftarrow \Delta_j$

Now the second simplex table (Table 5.10) is constructed as below :

Table 5·10

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	3 X_1	5 X_2	4 X_3	0 X_4	0 X_5	0 X_6	MIN RATIO (X_B/X_3)
$\rightarrow x_2$	5	8/3	2/3	1	0	1/3	0	0	—
$\leftarrow x_5$	0	14/3	– 4/3	0	[5]	-2/3	1	0	$\frac{14}{3}/5 \leftarrow$
x_6	0	29/3	5/3	0	4	– 2/3	0	1	$\frac{29}{3}/4$
$x_4 = x_1 = x_3 = 0$	$z = \frac{40}{3}$		1/3	0	–4* ↑	5/3	0 ↓	0	$\leftarrow \Delta_j$
					Incoming		Outgoing		

Now verify that

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = -3 + (5, 0, 0)(2/3, -4/3, 5/3) = 1/3$$

$$\Delta_3 = \mathbf{C_B X_3} - c_3 = -4 + (5, 0, 0)(0, 5, 4) = -4$$

$$\Delta_4 = \mathbf{C_B X_4} - c_4 = 0 + (5, 0, 0)(1/3, -2/3, -2/3) = 5/3 .$$

The key-element is found to be 5. Hence the vector to be removed from the basis matrix is $\mathbf{X_5}$. Thus proceeding exactly in the same manner, the remaining simplex tables are obtained (Tables 5.11 and 5.12).

Table 5·11

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{2}{3}/\frac{2}{3}$
$\rightarrow x_3$	4	14/15	– 4/15	0	1	– 2/15	1/15	0	—
$\leftarrow x_6$	0	89/15	[41/15]	0	0	– 2/15	– 4/5	1	$\frac{89}{15}/\frac{41}{15} \leftarrow$
$x_1 = x_5 = x_4 = 0$	$z = 256/15$		– 11/15* ↑	0	0	17/15	4/5	0 ↓	$\leftarrow \Delta_j$
			Incoming				Outgoing		

Table 5·12. Final Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO
x_2	5	50/41	0	1	0	15/41	8/41	– 10/41	
x_3	4	62/41	0	0	1	– 6/41	5/41	4/41	
x_1	3	89/41	1	0	0	– 2/41	– 12/41	15/41	
$x_4 = x_5 = x_6 = 0$	$z = \mathbf{C_B X_B} = 765/41$		0	0	0	45/41	24/41	11/41	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, the solution given by $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$, max $z = 765/41$, is optimal.

Example 6. *Minimize* $z = x_2 - 3x_3 + 2x_5$ *,subject to the constraints :*

$$3x_2 - x_3 + 2x_5 \leq 7, \; -2x_2 + 4x_3 \leq 12, \; -4x_2 + 3x_3 + 8x_5 \leq 10, \textit{ and } x_2, x_3, x_5 \geq 0 .$$

[JNTU (Mech.) 99; Kanpur 96; Madurai B.Sc. (Comp. Sc.) 92, (Appl. Math) 85; Kerala B.Sc. (Math.) 90]

Solution. Equivalently, max $z' = -x_2 + 3x_3 - 2x_5$ where $z' = -z$. Introducing x_1, x_4 and x_6 as slack variables, the constraint equations become :

$$x_1 + 3x_2 - x_3 + 0x_4 + 2x_5 + 0x_6 = 7$$

$$0x_1 - 2x_2 + 4x_3 + x_4 + 0x_5 + 0x_6 = 12$$

$$0x_1 - 4x_2 + 3x_3 + 0x_4 + 8x_5 + x_6 = 10.$$

Now proceeding as in above example the simplex computations are performed as follows :

Table 5·13

BASIC VARIABLES	C_B	$c_j \rightarrow$ / X_B	0 / X_1	–1 / X_2	3 / X_3	0 / X_4	–2 / X_5	0 / X_6	MIN RATIO (X_B/X_k)
x_1	0	7	1	3	–1	0	2	0	—
$\leftarrow x_4$	0	12	0	–2	[4]	1	0	0	12/4 $\leftarrow$
x_6	0	10	0	–4	3	0	8	1	10/3
		$z' = 0$	0	1	–3* ↑	0 ↓	2	0	$\leftarrow \Delta_j$
$\leftarrow x_1$	0	10	1	[5/2]	0	1/4	2	0	4 $\leftarrow$
$\rightarrow x_3$	3	3	0	–1/2	1	1/4	0	0	—
x_6	0	1	0	–5/2	0	–3/4	8	1	—
		$z' = 9$	0 ↓	–1/2* ↑	0	3/4	2	0	$\leftarrow \Delta_j$
x_2	–1	4	2/5	1	0	1/10	4/5	0	
x_3	3	5	1/5	0	1	3/10	2/5	0	
x_6	0	11	1	0	0	–1/2	10	1	
		$z' = 11$ or $z = -11$	1/5	0	0	4/5	12/5	0	$\leftarrow \Delta_j \geq 0$

Thus, optimal solution is : $x_2 = 4$, $x_3 = 5$, $x_5 = 0$, min. $z = -11$.

Alternative forms of Example 6 :

(i) *Min.* $z = x_1 - 3x_2 + 2x_3$, *subject to* $3x_1 - x_2 + 2x_3 \leq 7$, $-2x_1 + 4x_2 \leq 12$, $-4x_1 + 3x_2 + 8x_3 \leq 10$ *and* $x_1, x_2, x_3 \geq 0$.

(ii) *Min.* $z = x_2 - 3x_3 + 2x_5$, *subject to the constraints* :

$$x_1 + 3x_2 - x_3 + 2x_5 = 7, -2x_2 + 4x_3 + x_4 = 12, -4x_2 + 3x_3 + 8x_5 + x_6 = 10 \text{ and } x_1, x_2, \dots, x_6 \geq 0 .$$

Example 7 (Bounded Variables). *A manufacturer of three products tries to follow a policy of producing those which continue most to fixed cost and profit. However, there is also a policy of recognising certain minimum sales requirements currently, these are* :

Product :	X_1	X_2	X_3
Units per week :	20	30	60

There are three producing departments. The product times in hour per unit in each department and the total times available for each week in each department are :

Departments	*Time required per product in hours*			*Total hours available*
	X_1	X_2	X_3	
1	0·25	0·20	0·15	420
2	0·30	0·40	0·50	1048
3	0·25	0·30	0·25	529

The contribution per unit of product X_1 , X_2 , X_3 *is Rs.* 10·50, *Rs.* 9·00 *and Rs.* 8·00 *respectively. The company has scheduled 20 units of* X_1 , *30 units of* X_2 *and 60 units of* X_3 *for production in the following week, you are required to state :*

(i) *Whether the present schedule is an optimum one from a profit point of view and if it is not, what it should be;*

(ii) *The recommendations that should be made to the firm about their production facilities (following the answer to (i) above).*

Solution. The formulation of the problem is as follows :

Maximize $z = 10{\cdot}5X_1 + 9X_2 + 8X_3$, *subject to the constraints :*

$$0{\cdot}25X_1 + 0{\cdot}20X_2 + 0{\cdot}15X_3 \leq 420$$

$$0{\cdot}30X_1 + 0{\cdot}40X_2 + 0{\cdot}50X_3 \leq 1048$$

$$0{\cdot}25X_1 + 0{\cdot}30X_2 + 0{\cdot}25X_3 \le 529$$
$$0 \le X_1 \ge 20\,,\ 0 \le X_2 \ge 30\,,\ 0 \le X_3 \ge 60\,.$$

Since the company is already producting minimum of X_2 and X_3 it should, at least, produce maximum of X_1 limited by the first constraint. Lower bounds are specified in this problem, i.e., $X_1 \ge 20$, $X_2 \ge 30$, $X_3 \ge 60$. This can be handled quite easily by introducing the new variables x_1, x_2 and x_3 such that

$$X_1 = 20 + x_1, X_2 = 30 + x_2, X_3 = 60 + x_3\,.$$

Substituting for X_1, X_2 and X_3 in terms of x_1, x_2, x_3, the problem now becomes :

Maximize $z = 10{\cdot}5x_1 + 9x_2 + 8x_3 +$ constant, *subject to the constraints :* $0{\cdot}25x_1 + 0{\cdot}20x_2 + 0{\cdot}15x_3 \le 400$,

$0{\cdot}30x_1 + 0{\cdot}40x_2 + 0{\cdot}50x_3 \le 1000$, $0{\cdot}25x_1 + 0{\cdot}30x_2 + 0{\cdot}25x_3 \le 500$, *and* $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

The students may now proceed to find the optimal solution by simplex method in the usual manner.

Example 8. *For a company engaged in the manufacture of three products, viz. X, Y and Z, the available data are given below :*

Minimum Sales Requirement

Product:	X	Y	Z
Min. sales requirement per month:	10	20	30

Operations, Required Processing Times and Capacity

Operations	*Time (hrs.) required per item of*			*Total available hours per month*
	X	*Y*	*Z*	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180

Profit (Rs.) per unit

Product:	*X*	*Y*	*Z*
Profit (Rs.)/unit:	10	15	8

Find out the product-mix to maximize profit. **[C.A. (Nov.) 89]**

Solution. Let x, y and z denote the number of units produced per month for the products X, Y and Z, respectively.

Minimum sales requirements give the constraints : $x \ge 10$, $y \ge 20$, $z \ge 30$, where $x, y, z \ge 0$.

Operations, processing times and capacity lead to the following constraints :

$$x + 2y + 2z \le 200 \ldots\text{(i)} \qquad 2x + y + z \le 220 \ldots\text{(ii)} \qquad 3x + y + 2z \le 180 \ldots\text{(iii)}$$

The objective function is : Max. $\mathbf{P} = 10x + 15y + 8z$. Thus we have to solve the following problem :

Max. $\mathbf{P} = 10x + 15y + 8z$, *subject to* $x + 2y + 2z \le 200$, $2x + y + z \le 220$, $3x + y + 2z \le 180$, and $0 \le x \ge 10$, $0 \le y \ge 20$, $0 \le z \ge 30$.

Let us make the substitutions : $x = a + 10$, $y = b + 20$, $z = c + 30$, where $a, b, c \ge 0$.

Substituting these values in the objective function and constraints (i), (ii) and (iii), the problem becomes :

$$\textit{Max. } \mathbf{P} = 10a + 15b + 8c + 640, \ \textit{subject to,}$$
$$(a + 10) + 2(b + 20) + 2(c + 30) \le 200$$
$$2(a + 10) + (b + 20) + (c + 30) \le 220$$
$$3(a + 10) + (b + 20) + 2(c + 30) \le 180$$

where $a \ge 0, b \ge 0, c \ge 0$.

Solving this problem by simplex method we get the solution : $a = 10$, $b = 40$ and $c = 0$. Substituting these values, we find the original values :

$x = 10 + 10 = 20$, $y = 40 + 20 = 60$, $z = 0 + 30 = 30$, and the maximum value of objective function is given by $P =$ Rs. 1340.

The optimal product mix is to produce 20 units of X, 60 units of Y, and 30 units of Z to get a maximum profit of Rs. 1340.

Example 9. *Nooh's Boats makes three different kinds of boats. All can be made profitably in this company, but the company's monthly production is constrained by the limited amount of labour, wood and screws available each month. The director will choose the combination of boats that maximizes his revenue in view of the information given in the following table :*

Input	Row Boat	Canoe	Keyak	Monthly Available
Labour (Hours)	12	7	9	1.260 hrs.
Wood (Board feet)	22	18	16	19,008 board feet
Screws (Kg.)	2	4	3	396 Kg.
Selling price (in Rs.)	4,000	2,000	5,000	

(a) Formulate the above as a linear programming problem.

(b) Solve it by simplex method. From the optimal table of the solved linear programming problem, answer the following questions :

(c) How many boats of each type will be produced and what will be the resulting revenue ?

(d) Which, if any, of the resources are not fully utilized ? If so, how much of spare capacity is left ?

(e) How much wood will be used to make all of the boats given in the optimal solution ? **[C.A. (Nov.) 93]**

Solution. (a) Let x_1, x_2 and x_3 be the number of *Row Boats, Canoe* and *Keyak* made every month. The linear programming model can be formulated as follows :

Max. Revenue $z = 4{,}000x_1 + 2{,}000x_2 + 5{,}000x_3$, *subject to*

$12x_1 + 7x_2 + 9x_3 \le 1260,\ 22x_1 + 18x_2 + 16x_3 \le 19008,\ 2x_1 + 4x_2 + 3x_3 \le 396$ and $x_1, x_2, x_3, \ge 0$.

(b) Adding slack variables s_1, s_2, s_3, the above formulated problem becomes

Max. $z = 4000x_1 + 2000x_2 + 5000x_3 + 0s_1 + 0s_2 + 0s_3$, *subject to :*

$12x_1 + 7x_2 + 9x_3 + s_1 = 1260,\ 22x_1 + 18x_2 + 16x_3 + s_2 = 19008,\ 2x_1 + 4x_2 + 3x_3 + s_3 = 396$, and $x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$.

The starting solution and subsequent simplex tables are given below :

		$c_j \rightarrow$	4000	2000	5000	0	0	0	
Basic Variables	Prog. C_B	Qty X_B	X_1	X_2	X_3	S_1	S_2	S_3	Replacement Ratio Min (X_B/X_K)
s_1	0	1,260	12	7	9	1	0	0	1260/9
s_2	0	19,008	22	18	16	0	1	0	19008/16
s_3	0	396	2	4	← [3]	0	0	1	396/3 ←
	$z = 0$		−4000	−2000	−5000↑	0	0	0↓	← Δ_j (NER)
s_1	0	72	[6]	−5	0	1	0	−3	12 ←
s_2	0	16,896	34/3	−10/.3	0	0	1	−16/3	1491
x_3	5000	132	2/3	4/3	1	0	0	1/3	198
	$z = 660000$		−2000/3	14,000/3↑	0	0↓	0	5000/3	← Δ_j
x_1	4000	12	1	−5/6	0	1/6	0	−1/2	
s_2	0	16,760	0	55/9	0	−17/9	1	1/3	
x_3	5000	124	0	17/9	1	−1/9	0	2/3	
	$z = 6{,}68{,}000$		0	37,000/9	0	1000/9	0	4000/3	← Δ_j

Since all $\Delta_j \ge 0$, the optimal solution is given by $x_1 = 12$, $x_2 = 0$ and $x_3 = 124$.

(c) The company should produce 12 *Row boats* and *124 Kayak boats* only. The maximum revenue will be Rs. 6,68,000.

(d) Wood is not fully utilized. Its share capacity is 16,760 board feet.

(e) The total wood used to make all of the boats given by the optimum solution is

$$= 22 \times 12 + 16 \times 124 = 2{,}248 \text{ board feet.}$$

EXAMINATION PROBLEMS

Solve the following problems by simplex method :

1. Max. $z = 5x_1 + 3x_2$, subject to

$3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0$. **[JNTU (B. Tech. III) 2003; Kanpur B.Sc. (ii) 2003]**

[**Ans.** $x_1 = 20/19$, $x_2 = 45/19$, Max. $z = 235/19$]

2. Max. $z = 7x_1 + 5x_2$, subject to

$-x_1 - 2x_2 \ge -6$

$4x_1 + 3x_2 \le 12$,

$x_1, x_2 \ge 0$ **[Meerut (IPM) 91]**

[**Ans.** $x_1 = 3$, $x_2 = 0$, Max. $z = 21$]

3. Max. $z = 5x_1 + 7x_2$, subject to
$x_1 + x_2 \le 4$
$3x_1 - 8x_2 \le 24$
$10x_1 + 7x_2 \le 35$
and $x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 0, x_2 = 4$, *max.* $z = 28$]

4. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 40$
$x_1 + x_2 \le 24$
$2x_1 + 3x_2 \le 60$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 16, x_2 = 8, z^* = 64$]

5. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 5$
$x_1 + x_2 \le 3$
$x_1, x_2 \ge 0$
and $x_1, x_2 \ge 0$. **[I.A.S (Main) 91]**
[**Ans.** $x_1 = 6, x_2 = 12, z^* = 60$]

6. Max. $z = 2x_1 + 4x_2$, subject to
$2x_1 + 3x_2 \le 48$
$x_1 + 3x_2 \le 42$
$x_1 + x_2 \le 21$
and $x_1, x_2 \ge 0$
[**Ans.** Solution is unbounded]

7. Max. $z = 3x_1 + 4x_2$, subject to
$x_1 - x_2 \le 1$
$-x_1 + x_2 \le 2$
$x_1, x_2 \ge 0$.
[**Ans.** Sol. is unbounded]

8. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 10$
$x_1 + 3x_2 \le 6$
$x_1, x_2 > 0$
[**Ans.** $x_1 = 24/5, x_2 = 2/5, z^* = 76/5$]

9. Max. $z = 2x_1 + 5x_2$, subject to
$x_1 + 3x_2 \le 3$
$3x_1 + 2x_2 \le 6$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2, x_2 = 0, z^* = 4$]

10. Max. $z = 3x_1 + 5x_2$, subject to
$3x_1 + 2x_2 \le 18$
$x_1 \le 4$
$x_2 \le 6$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2, x_2 = 6, z^* = 36$]

11. Max. $z = 2x_1 + x_2$, subject to
$x_1 + 2x_2 \le 10$
$x_1 + x_2 \le 6$
$x_1 - x_2 \le 2$
$x_1 - 2x_2 \le 1$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 4, x_2 = 2, z^* = 10$]

12. Max. $z = 2x + 5y$, subject to
$x + y \le 600$
$0 \le x \le 400$
$0 \le y \le 300$
[**Ans.** Two Iterations.
$x = 300, y = 300$, max $z = 2100$]

13. Max. $z = x_1 - x_2 + 3x_3$, subject to
$x_1 + x_2 + x_3 \le 10$
$2x_1 - x_3 \le 2$
$2x_1 - 2x_2 + 3x_3 \le 0$
and $x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0, x_2 = 6, x_3 = 4, z^* = 6$]

14. Max. $z = x_1 + x_2 + x_3$, subject to
$4x_1 + 5x_2 + 3x_3 \le 15$
$10x_1 + 7x_2 + x_3 \le 12$
and $x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0, x_2 = 0, x_3 = 5, z^* = 5$]

15. Max. $z = 8x_1 + 19x_2 + 7x_3$,
subject to
$3x_1 + 4x_2 + x_3 \le 25$
$x_1 + 3x_2 + 3x_3 \le 50$
$x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 7/3, x_2 = 9, x_3 = 0$]

16. Max. $z = x_1 + x_2 + 3x_3$,
subject to
$3x_1 + 2x_2 + x_3 \le 3$
$2x_1 + x_2 + 2x_3 \le 2$
$x_1, x_2, x_3 \ge 0$.
[VTU (BE common) 2002]
[**Ans.** $x_1 = 0, x_2 = 0, x_3 = 1, z^* = 3$]

17. Max. $z = 4x_1 + 3x_2 + 4x_3 + 6x_4$,
subject to
$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80$
$2x_1 + 2x_3 + x_4 \le 60$
$3x_1 + 3x_2 + x_3 + x_4 \le 80$
$x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 280/13, x_2 = 0$,
$x_3 = 20/13, x_4 = 180/13$
$z^* = 2280/13$].

18. Max. $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$.
subject to
$x_1 + x_2 + x_3 + x_4 \le 15$
$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$
$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100$
$x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 50/7, x_2 = 0, x_3 = 55/7$,
$x_4 = 0, z^* = 695/7$]

19. Max. $z = 2x_1 + 4x_2 + x_3 + x_4$,
subject to
$2x_1 + x_2 + 2x_3 + 3x_4 \le 12$,
$3x_1 + 2x_3 + 2x_4 \le 20$,
$2x_1 + x_2 + 4x_3 \le 16$,
$x_1, x_2, x_3, x_4 \ge 0$. **[JNTU (MCA) 2004]**
[**Ans.** $x_1 = x_3 = x_4 = 0, x_2 = 12$,
max. $z = 48$. On iteration only]

20. Max. $z = 5x_1 + 3x_2$,
subject to the constraints :
$x_1 + x_2 \le 2$
$5x_1 + 2x_2 \le 10$
$3x_1 + 8x_2 \le 12$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2, x_2 = 0$, max. $z = 10$,
one iteration only]

21. Max. $z = 8x_1 + 11x_2$
subject to the constraints :
$3x_1 + x_2 \le 7, x_1 + 3x_2 \le 8, x_1, x_2 \ge 0$.
[**Ans.** Two iterations.
$x_1 = 13/8, x_2 = 17/8$, max $z = 291/8$]

22. Max. $z = 10x_1 + x_2 + 2x_3$, subject to the constraints
$x_1 + x_2 - 3x_3 \le 10, 4x_1 + x_2 + x_3 \le 20, x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 5, x_2 = 0, x_3 = 0$, max. $z = 50$]

23. Max. $z = 2x_1 + 4x_2 + x_3 + x_4$, subject to the constraints : $x_1 + 3x_2 + x_4 \le 4, 2x_1 + x_2 \le 3, x_2 + 4x_3 + x_4 \le 3; x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 1, x_2 = 1, x_3 = 1/2, x_4 = 0$, max. $z = 13/2$]

24. Max. $z = 10x_1 + 6x_2$, subject to the constraints :
$x_1 + x_2 \le 2, 2x_1 + x_2 \le 4, 3x_1 + 8x_2 \le 12$, and
$x_1, x_2 \ge 0$.
[**Ans.** One iteration only.
$x_1 = 2, x_2 = 0$, max. $z = 20$]

25. Max. $z = 107x_1 + x_2 + 2x_3$, subject to the constraints :
$14x_1 + x_2 - 6x_3 + 3x_4 = 7, 16x_1 + 1/2x_2 - 6x_3 \le 5, 3x_1 - x_2 - x_3 \le 0$.
$x_1, x_2, x_3, x_4 \ge 0$.
[**Hint.** Divide the first equation by 3 (coefficient of x_4) and then treat x_4 as the slack variable]. [**Ans.** Unbounded solution].

26. Explain the simplex method by carrying out one iteration in the following problem :
Max. $z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$, subject to the constraints :

$x_1 + 2x_2 + 2x_3 + x_4 = 8,\ 3x_1 + 4x_2 + x_3 + x_5 = 7$ and $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$.
[**Ans.** One iteration only, $x_1 = x_2 = x_4 = 0,\ x_3 = 4,\ x_5 = 3$, max. $z = 15$]

27. Max. $z = 3x_1 + 2x_2 - 2x_3$
subject to the constraints :
$x_1 + 2x_2 + 2x_3 \leq 10$
$2x_1 + 4x_2 + 3x_3 \leq 15$
$x_1,\ x_2 < x_3 \geq 0$
[**Ans.** One iteration only.
$x_1 = 15/2,\ x_2 = x_3 = 0$, max. $z = 45/2$]

28 Max. $z = 30x_1 + 23x_2 + 29x_3$
subject to the constraints :
$6x_1 + 5x_2 + 3x_3 \leq 52$
$6x_1 + 2x_2 + 5x_3 \leq 14$.
$x_1,\ x_2,\ x_3 \geq 0$.
[**Ans.** Two iterations. $x_1 = 0,\ x_2 = 7,\ x_3 = 0$, max. $z = 161$]

29. Max. $z = 7x_1 + x_2 + 2x_3$,
subject to the constraints :
$x_1 + x_2 - 2x_3 \leq 10$
$4x_1 + x_2 + x_3 \leq 20$
$x_1,\ x_2,\ x_3 \geq 0$.
[**Ans.** Two iterations. $x_1 = x_2 = 0,\ x_3 = 20$
max. $z = 40$]

30. Max. $R = 2x - 3y + z$
subject to the constraints :
$3x + 6y + z \leq 6$
$4x + 2y + z \leq 4$
$x - y + z \leq 3$
$x,\ y,\ z \geq 0$
[**Ans.** Two iterations. $x = 1/3$,
$y = 0,\ z = 8/3$, max. $R = 10/3$]

31. Max. $R = 2x + 4y + 3z$
subject to the constraints :
$3x + 4y + 2z \leq 60$
$2x + y + 2z \leq 40$
$x + 3y + 2z \leq 80$
$x,\ y,\ z \geq 0$.
[**Ans.** Two iterations. $x = 0,\ y = 20/3$,
$z = 50/7$, max. $R = 250/3$.]

32. Max. $z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$
subject to the constraints :
$x_1 + x_4 + 6x_6 = 9$
$3x_1 + x_2 - 4x_3 + 2x_6 = 2$
$x_1 + 2x_3 + x_5 + 2x_6 = 6$
$x_i \geq 0,\ i = 1, 2, 3, 4, 5, 6$.
[**Ans.** One iteration. $x_1 = 2/3,\ x_2 = x_3 = 0$,
$x_4 = 25/3,\ x_5 = 16/3,\ x_6 = 0$, max $z = 43/3$]

33. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yield a profit of Rs. 30. An acre of wheat cost Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans cost Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8,000 man-days of work, how masny acres should be allocated to each crop to maximize profit ? **[Jammu Univ. (MBA) Feb. 96]**
[**Hint.** Formulation of the problem is :
Max. $z = 30x_1 + 40x_2 + 20x_3$, s.t.
$10x_1 + 12x_2 + 7x_3 \geq 10{,}000;\ 7x_1 + 10x_2 + 8x_3 \leq 8{,}000$
$x_1 + x_2 + x_3 \leq 1{,}000;\ x_1, x_2, x_3 \geq 0$.]
[**Ans.** Acreage for carn, wheat and soyabeans are 250, 625 and respectively with max. profit of Rs. 32,500]

5·5. ARTIFICIAL VARIABLE TECHNIQUES

5·5-1. Two Phase Method **[Garhwal 97; Kanpur (B.Sc.) 90; Rohil. 90]**

Linear programming problems, in which constraints may also have '$\geq$' and '$=$' signs after ensuring that all b_i are ≥ 0, are considered in this section. In such problems, basis matrix is not obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable, called, the ***artificial variable.*** These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. Artificial variables can be eliminated from the simplex table as and when they become zero (non-basic). The process of eliminating artificial variables is performed in ***Phase I*** of the solution, and ***Phase II*** is used to get an optimal solution. Since the solution of the LP problem is completed in two phases, it is called ***'Two Phase Simplex Method'*** due to **Dantzig, Orden** and **Wolfe.**

Remarks :

1. **The objective of Phase I is to search for a B.F.S. to the given problem It ends up either giving a B.F.S. or indicating that the given L.P.P. has no feasible solution at all.**
2. **The B.F.S. obtained at the end of Phase 1 provides a starting B.F.S. for the given L.P.P. Phase II is then just the application of simplex method to move towards optimality.**
3. **In Phase II, care must be taken to ensure that an artificial variable is never allowed to become positive, if were present in the basis. Moreover, whenever some artifical variable happens to leave the basis, its column must be deleted from the simplex table altogether.**

Q. 1. Explain the term 'Artificial variable' and its use in linear programming.
2. What do you mean by two phase-method in linear programming problems, why it is used ?

This technique is well explained by the following example.

Example 10. *Solve the problem : Minimize* $z = x_1 + x_2$, *subject to* $2x_1 + x_2 \geq 4$, $x_1 + 7x_2 \geq 7$, *and* $x_1 , x_2 \geq 0$. **[Kanpur (B.Sc.) 03; Delhi B.Sc. (Math.) 91, 88; Bharthidasan B.Sc. (Math.) 90; VTU (BE, Common) Aug. 02]**

Solution. First convert the problem of minimization to maximization by writing the objective function as :

$$\text{Max} \quad (-z) = -x_1 - x_2 \quad \text{or} \quad \text{Max. } z' = -x_1 - x_2 \text{, where } z' = -z.$$

Since all b_i's (4 and 7) are positive, the 'surplus variables' $x_3 \geq 0$ and $x_4 \geq 0$ are introduced, then constraints become :

$$2x_1 + x_2 - x_3 = 4$$
$$x_1 + 7x_2 - x_4 = 7 .$$

But the basis matrix **B** would not be an identity matrix due to negative coefficients of x_3 and x_4 . Hence the starting basic feasible solution cannot be obtained.

On the other hand, if so-called 'artificial variables' $a_1 \geq 0$ and $a_2 \geq 0$ are introduced, the constraint equations can be written as

$$2x_1 + x_2 - x_3 + a_1 = 4$$
$$x_1 + 7x_2 - x_4 + a_2 = 7 .$$

It should be noted that $a_1 < x_3$, $a_2 < x_4$, otherwise the constraints of the problem will not hold.

Phase I. Construct the first table (*Table 5·14*) where A_1 and A_2 denote the artificial column-vectors corresponding to a_1 and a_2, respectively.

Table 5·14

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	4	2	1	−1	0	1	0
a_2	7	1	[7]	0	−1	0	1
			↑	×	×	×	↓

Now remove each artificial column vector A_1 and A_2 from the basis matrix. To remove vector A_2 first, select the entering vector either X_1 or X_2 , being careful to choose any one that will yield a non-negative (feasible) revised solution. Take the vector X_2 to enter the basis matrix. It can be easily verified that if the vector A_2 is entered in place of X_1 , the resulting solution will not be feasible. Thus transformed table (Table 5·15) is obtained.

Table 5·15

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	3	13/7	0	−1	[1/7]	1	−1/7
x_2	1	1/7	1	0	−1/7	0	1/7
					↑	↓	

(Delete column A_2 for ever at this stage)

This table gives the solution : $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, $a_1 = 3$, $a_2 = 0$. When the artificial variable a_2 becomes zero (non-basic), we forget about it and never consider the corresponding vector A_2 again for re-entry into the basis matrix.

Similarly, remove A_1 from the basis matrix by introducing it in place of X_4 by the same method. Thus Table 5·16 is obtained.

Table 5·16

BASIC VARAIBLES	X_B	X_1	X_2	X_3	X_4	A_1
x_4	21	13	0	−7	1	7
x_2	4	2	1	−1	0	1

(Delete column A_1 for ever at this stage)

This table gives the solution : $x_1 = 0, x_2 = 4$, $x_3 = 0$, $x_4 = 21$, $a_1 = 0$. Since the artificial variable a_1 becomes zero (non-basic), so drop the corresponding column A_1 from this table. Thus, the solution ($x_1 = 0$, $x_2 = 4, x_3 = 0$, $x_4 = 21$) is the basic feasible solution and now usual simplex routine can be started to obtain the required optimal solution.

Phase II. Now in order to test the starting above solution for optimality, construct the starting simplex Table 5.17

Table 5·17

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	Min. Ratio (X_B/X_1)
$c_j \rightarrow$			−1	−1	0	0	
$\leftarrow x_4$	0	21	← [13] ↑	0	−7	1	21/13 ←
x_2	−1	4	2	1	−1	0	4/2
	$z' = C_B X_B$	$= -4$	−1 ↑	0	1	0 ↓	$\leftarrow \Delta_j$

Compute $\Delta_1 = -1$, $\Delta_3 = 1$

Key element 13 indicates that X_4 should be removed from the basis matrix. Thus, by usual transformation method Table 5·18 is formed.

Table 5·18

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	MIN. RATIO COLUMN
$c_j \rightarrow$			−1	−1	0	0	
$\rightarrow x_1$	−1	21/13	1	0	−7/13	1/13	
x_2	−1	10/13	0	1	1/13	−2/13	
	$z' = -31/13$		0	0	6/13	1/13	$\leftarrow \Delta_j \geq 0$

Also, verify that

$$\Delta_3 = C_B X_3 - c_3 = (-1, -1)(-7/13, 1/13) = 6/13$$

$$\Delta_4 = C_B X_4 - c_4 = (-1, -1)(1/13, -2/13) = 1/13 .$$

Since all $\Delta_j \geq 0$, the required optimal solution is :

$$x_1 = 21/13 , x_2 = 10/13 \text{ and min. } z = 31/13 \text{ (because } z = -z') .$$

5.5·2. Simple Way for Two-Phase Simplex Method

Phase I : Table 5·19

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	4	2	1	−1	0	1	0
$\leftarrow a_2$	7	1	[7] ↑	0	−1	0	1 ↓
$\leftarrow a_1$	3	13/7	0	−1	[1/7]	1	−1/7
$\rightarrow x_2$	1	1/7	1	0	−1/7 ↑	0 ↓	1/7
$\rightarrow x_4$	21	13	0	−7	1	7	×
x_2	4	2	1	−1	0	1	×

Thus, initial basic feasible solution is : $x_1 = 0$, $x_2 = 4$, $x_3 = 0$, $x_4 = 21$. Now start to improve this solution in Phase II by usual simplex method.

Note.

1. Remove the artificial vector A_2 and insert it anywhere such that X_B remains feasible (≥ 0) .
2. As soon as A_2 is removed from the basis by matrix transformation or otherwise, delete A_2 for ever.
3. Similar process is adopted to remove other artificial vectors one by one from the basis.
4. Purpose of introducing artificial vectors is only to provide an initial basic feasible solution to start with simplex method in Phase II. So, as soon as the artificial variables become non-basic (*i.e.* zero), delete artificial vectors to enter Phase II.
5. Then, start Phase II, which is exactly the same as original simplex method.

Phase II. Table 5·20

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	MIN. RATIO (X_B/X_k)
$c_j \rightarrow$			-1	-1	0	0	
$\leftarrow x_4$	0	21	[13]	0	7	1	$21/13 \leftarrow$
x_2	-1	4	2	1	-1	0	4/2
	$z' = -4$		-1^* ↑	0	1	0 ↓	$\leftarrow \Delta_j$
$\rightarrow x_1$	-1	21/13	1	0	$-7/13$	1/13	
x_2	-1	10/13	0	1	1/10	2/13	
	$z' = -31/13$		0	0	6/13	1/13	$\leftarrow \Delta_j \geq 0$

Thus, the desired solution is obtained as : $x_1 = 21/13$, $x_2 = 10/13$, max. $z = 31/13$.

5.5–3. *Alternative Approach* of Two-phase Simplex Method

The two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase I. In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1. Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function.

Step 2. Construct the auxiliary linear programming problem in which the new objective function z^* is to be maximized subject to the given set of constraints.

Step 3. Solve the auxiliary problem by simplex method until either of the following three possibilities do arise :

(i) Max $z^* < 0$ and at least one artificial vector appear in the optimum basis at a positive level. In this case given problem does not possess any feasible solution.

(ii) Max $z^* = 0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to *Phase–II.*

(iii) Max $z^* = 0$ and no artificial vector appears in the optimum basis. In this case also proceed to *Phase–II.*

Phase II. Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. That is, simplex method is applied to the modified simplex table obtained at the end of *Phase–I*, until an optimum basic feasible solution (if exists) has been attained.The artificial variables which are non-basic at the end of *Phase–I* are removed.

Q. 1. What are artificial variables ? Why do we need them ? Describe briefly the two-phase method of solving a L.P. problem with artificial variables. **[Meerut M.Sc. (Math.) 93]**

2. What do you mean by two phase method for solving a given L.P.P. ? Why is it used ?

3. Explain steps in solving a linear programming problem by two-phase method.

The following examples will make the *alternative* two-phase method clear.

Example 11. *Use two-phase simplex method to solve the problem : Minimize* $z = x_1 - 2x_2 - 3x_3$, *subject to the constraints* :– $2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$, *and* $x_1, x_2, x_3 \geq 0$, **[Meerut (Maths) 91]**

Solution. First convert the objective function into maximization form :

$$\text{Max } z' = -x_1 + 2x_2 + 3x_3 , \text{ where } z' = -z .$$

Introducing the artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the constraints of the given problem become,

$$-2x_1 + x_2 + 3x_3 + a_1 = 2$$
$$2x_1 + 3x_2 + 4x_3 + a_2 = 1$$
$$x_1, x_2, x_3, a_1, a_2 \geq 0.$$

Phase I. Auxiliary L.P. problem is : *Max.* $z'^* = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$ *subject to above given constraints.*

The following solution table is obtained for auxiliary problem.

Table 5·21

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	A_1	A_2	MIN. RATIO (X_B/X_k)
$c_j \to$			0	0	0	−1	−1	
a_1	−1	2	−2	1	3	1	0	2/3
$\leftarrow a_2$	−1	1	2	3	[4]	0	1	1/4 ←
	$z'^* = -3$		0	−4	−7* ↑	0	0 ↓	← Δ_j
a_1	−1	5/4	−7/2	−5/4	0	1	−3/4	
$\to x_3$	0	1/4	1/2	3/4	1	0	1/4	
	$z'^* = -5/4$		7/4	5/4	0	0	3/4	← $\Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution to the auxiliary L.P.P. has been attained. But at the same time max. z'^* is negative and the artificial variable a_1 appears in the basic solution at a positive level. Hence the original problem does not possess any feasible solution. Here there is no need to enter *Phase II.*

Example 12. *Use two-phase simplex method to solve the problem :*

Minimize $z = 15/2\ x_1 - 3x_2$, *subject to the constraints* :

$$3x_1 - x_2 - x_3 \geq 3, \quad x_1 - x_2 + x_3 \geq 2, \text{ and } x_1, x_2, x_3 \geq 0.$$

Solution. Convert the objective function into the maximization form : Maximize. $z' = -15/2\ x_1 + 3x_2$.

Introducing the surplus variables $x_4 \geq 0$ and $x_5 \geq 0$, and artificial variables $a_1 \geq 0$, $a_2 \geq 0$, the constraints of the given problem become

$$3x_1 - x_2 - x_3 - x_4 + a_1 = 3$$
$$x_1 - x_2 + x_3 - x_5 + a_2 = 2$$
$$x_1, x_2, x_3, x_4, a_1, a_2 \geq 0.$$

Phase I. Assigning a cost −1 to artificial variables a_1 and a_2 and cost 0 to all other variables, the new objective function for auxiliary problem becomes : Max. $z'^* = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 - 1a_1 - 1a_2$, subject to the above given constraints.

Now apply simplex method in usual manner, (see *Table 5·22*).

Phase I : Table 5·22

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	A_2	MIN RATIO (X_B/X_k)
$c_j \to$			0	0	0	0	0	−1	−1	
$\leftarrow a_1$	−1	3	[3]	−1	−1	−1	0	1	0	3/3 ←
a_2	−1	2	1	−1	1	0	−1	0	1	2/1
	$z'^* = -5$		−4* ↑	2	0	1	1	0 ↓	0	← Δ_j
$\to x_1$	0	1	1	−1/3	−1/3	−1/3	0	1/3	0	—
$\leftarrow a_2$	−1	1	0	−2/3	[4/3]	1/3	−1	1/3	1	3/4 ←
	$z'^* = -1$		0	2/3	−4/3* ↑	−1/3	1	2/3	0 ↓	← Δ_j
x_1	0	5/4	1	−1/2	0	−1/4	−1/4	1/4	1/4	
x_3	0	3/4	0	−1/2	1	1/4	−3/4	1/4	3/4	
	$z'^* = 0$		0	0	0	0	0	1	1	← $\Delta_j \geq 0$

Since all $\Delta_j \geq 0$ and no artificial variable appears in the basis, an optimum solution to the auxiliary problem has been attained.

Phase 2. In this phase, now consider the actual costs associated with the original variables, the objective function thus becomes : Max. $z' = -15/2\,x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$

Now apply simplex method in the usual manner.

Phase 2 : Table 5·23

		$c_j \rightarrow$	$-15/2$	3	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	MIN RATIO (X_B/X_k)
x_1	$-15/2$	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	
x_3	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	
		$z' = -75/8$	0	$3/4$	0	$15/8$	$15/8$	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been attained.

Hence optimum solution is : $x_1 = 5/4$, $x_2 = 0$, $x_3 = 3/4$, min $z = 75/8$.

EXAMINATION PROBLEMS

Solve the following LP problems by two-phase method :

1. Max. $z = 3x_1 - x_2$
   subject to the constraints :
   $2x_1 + x_2 \geq 2$
   $x_1 + 3x_2 \leq 2$
   $x_2 \leq 4$
   and $x_1, x_2 \geq 0$.
   [**Ans.** $x_1 = 2$, $x_2 = 0$ Max $z = 6$]

2. Max. $z = 5x_1 + 8x_2$
   subject to the constraints :
   $3x_1 + 2x_2 \geq 3$
   $x_1 + 4x_2 \geq 4$
   $x_1 + x_2 \leq 5$
   and $x_1, x_2 \geq 0$.
   [JNTU (Mech. & Prod.) 2004]
   [**Ans.** $x_1 = 0$, $x_2 = 5$, max. $z = 40$]

3. Max $z = x_1 + 1.5x_2 + 2x_3 + 5x_4$
   with the conditions :
   $3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$
   $2x_1 + x_2 + x_3 + 5x_4 \leq 4$
   $2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$
   $x_1 + 3x_2 - 4x_3 + 3x_4 = 0$
   $x_i\,(i = 1, 2, 3, 4) \geq 0$
   [**Ans.** $x_1 = 1.2$, $x_2 = 0$, $x_3 = 0.9$
   $x_4 = 0$, max. $z = 19.8$]

4. Minimize $z = x_1 - 2x_2 - 3x_3$, subject to
   $-2x_1 + x_2 + 3x_3 = 2$
   $2x_1 + 3x_2 + 4x_3 = 1$,
   $x_j \geq 0$, $j = 1, 2, 3$,
   [**Ans.** Here all $\Delta_j \geq 0$, but at the same time artificial variable a_1 appears in the basis. Hence the given LPP has no feasible solution]

5. Max. $z = 3x_1 + 2x_2 + x_3 + 4x_4$
   subject to
   $4x_1 + 5x_2 + x_3 - 3x_4 = 5$
   $2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$
   $x_1 + 4x_2 + 2.5x_3 - 4x_4 = 6$
   $x_1, x_2, x_3 \geq 0$
   [**Ans.** No solution]

6. Max $z = 5x_1 - 2x_2 + 3x_3$
   subject to
   $2x_1 + 2x_2 - x_3 \geq 2$
   $3x_1 - 4x_2 \leq 3$
   $x_2 + 3x_3 \leq 5$
   $x_1, x_2, x_3, x_4 \geq 0$.
   [AIMS (BE Ind.) Bang. 2002]
   [**Ans.** $x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$,
   max. $z = 85/3$]

7. Max. $z = 2x_1 + 3x_2 + 5x_3$,
   subject to the constraints :
   $3x_1 + 10x_2 + 5x_3 \leq 15$,
   $x_1 + 2x_2 + x_3 \geq 4$,
   $33x_1 - 10x_2 + 9x_3 \leq 33$,
   $x_1, x_2, x_3 \geq 0$.

8. Max. $500\,x_1 + 1400\,x_2 + 900x_3$,
   subject to,
   $x_1 + x_2 + x_3 = 100$
   $12x_1 + 35x_2 + 15x_3 \geq 25$
   $8x_1 + 3x_2 + 4x_3 \geq 6$;
   $x_1, x_2, x_3 \geq 0$. **[Meerut (MA) 93]**

   [**Ans.** There does not exist any feasible solution, because artificial variable is not removed in the problem]

9. A firm has an advertising budget of Rs. 7,20,000. It wishes to allocate this budget to two media : magazines and televisions, so that total exposure is maximized. Each page of magazine advertising is estimated to result in 60,000 exposures, whereas each spot on television is estimated to result in 1,20,000 exposures. Each page of magazine advertising costs Rs. 9,000 and each spot on television costs Rs. 12,000. An additional condition that the firm has specified is that at least two pages of magazine advertising be used and at least 3 spots on television. Determine the optimum media-mix for this firm.
   [**Hint.** The problem is : **[Delhi (MBA) 97]**
   Max. $z = 60{,}000x_1 + 12{,}000x_2$ s.t.
   $9{,}000x_1 + 12{,}000x_2 \leq 7{,}20{,}000$, $x_1 \geq 2$, $x_2 \geq 3$, $x_1, x_2 \geq 0$,
   where x_1 = no. of pages of magazine
   x_2 = no. of spots on television]
   [**Ans.** $x_1 = 2$, $x_2 = 58{\cdot}5$ and max. $z = 7{,}14{,}000$]

5.5-4 Big-*M*-Method (Charne's Penalty Method) [Kanpur (B.Sc.) 92, 91]

Computational steps of big-*M*-method are as stated below :

Step 1. Express the problem in the standard form.

Step 2. Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type ($\geq$) and '='. When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists). On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by $-M$ for maximization problems ($+M$ for minimization problems), where $M > 0$.

Step 3. In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

Q. 1. Explain the use of Big-*M*-method in solving L.P.P. What are its characteristics ?

Example 13. *Solve by using big-M method the following linear programming problem :*

Max. $z = -2x_1 - x_2$, *subject to* $3x_1 + x_2 = 3, 4x_1 + 3x_2 \geq 6, x_1 + 2x_2 \leq 4$, *and* $x_1, x_2 \geq 0$.

[JNTU (B. Tech.) 2003]

Solution.

Step 1. Introducing slack, surplus and artificial variables, the system of constraint equations become :

$$3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - x_3 + a_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$

which can be written in the matrix form as :

$$\begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 & \mathbf{X}_4 & \mathbf{A}_1 & \mathbf{A}_2 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

Step 2. Assigning the large negative price $-M$ to the artificial variables a_1 and a_2, the objective function becomes : Max. $z = -2x_1 - x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2$.

Step 3. Construct starting simplex table (Table 5·24)

Starting Simplex Table 5·24

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$	MIN. RATIO $(\mathbf{X_B/X_1})$
$\leftarrow a_1$	$-M$	3	[3]	1	0	0	1	0	$3/3 \leftarrow$
a_2	$-M$	6	4	3	-1	0	0	1	6/4
x_4	0	4	1	2	0	1	0	0	4/1
		$z = -9M$	$(2-7M)$ $\uparrow$	$(1-4M)$	M	0	0 $\downarrow$	0	$\leftarrow \Delta_j$

To apply optimality test, compute

$\Delta_1 = \mathbf{C_B X_1} - c_1 = (-M, -M, 0)(3, 4, 1) - (-2) = 2 + (-3M - 4M + 0) = 2 - 7M$

$\Delta_2 = \mathbf{C_B X_2} - c_2 = (-M, -M, 0)(1, 3, 2) - (-1) = 1 + (-M - 3M + 0) = 1 - 4M$

$\Delta_3 = \mathbf{C_B X_3} - c_3 = (-M, -M, 0)(0, -1, 0) + 0 = M$

$\therefore \quad \Delta_k = \min [\Delta_1, \Delta_2, \Delta_3] = \min [2 - 7M, 1 - 4M, M] = \Delta_1$. Therefore, $\mathbf{X}_1$ will be entered.

Using minimum ratio rule, find the key element 3 which indicates that $\mathbf{A}_1$ should be removed. Now the transformed table (*Table 5·25*) is obtained in usual manner.

First Improved Table 5·25

BASIC VARIABLES	C_B	X_B	X_1 (−2)	X_2 (−1)	X_3 (0)	X_4 (0)	A_1 ($-M$)	A_2 ($-M$)	MIN RATIO (X_B/X_2)
$\rightarrow x_1$	−2	1	1	1/3 ↑	0	0	1/3	0	$1/\frac{1}{3}$
$\leftarrow a_2$	$-M$	2	0	← [5/3]	1	0	4/3	1	$2/\frac{5}{3}$ ←
x_4	0	3	0	5/3	0	1	−1/3	0	$3/\frac{5}{3}$
	$z=-2-2M$		0	$(1-5M)/3$ ↑	M	0	$(-2+7M)/3$	0 ↓	$\leftarrow \Delta_j$

Again compute, $\Delta_2 = \mathbf{C_B X_2} - c_2 = (-2, -M, 0)(1/3, 5/3, 5/3) + 1 = (1-5M)/3$, and similarly, $\Delta_3 = M$, $\Delta_5 = (-2+7M)/3$.

Since *minimum* Δ_j *rule and minimum ratio rule* decide the key element 5/3, so enter $\mathbf{X_2}$ and remove $\mathbf{A_2}$. Therefore, the second improved table (Table 5.26) is formed.

Table 5·26

BASIC VARIABLES	C_B	X_B	X_1 (−2)	X_2 (−1)	X_3 (0)	X_4 (0)	A_1 ($-M$)	A_2 ($-M$)	MIN. RATIO
x_1	−2	3/5	1	0	1/5	0	3/5	−1/5	
x_2	−1	6/5	0	1	−3/5	0	−4/5	3/5	
x_4	0	1	0	0	1	1	1	−1	
	$z = \mathbf{C_B X_B} = -12/5$		0	0	1/5	0	$M-2/5$	$M-1/5$	$\leftarrow \Delta_j \geq 0$

To test the solution for optimality, compute

$$\Delta_3 = \mathbf{C_B X_3} - c_3 = (-2, -1, 0)(1/5, -3/5, 1) - 0 = 1/5$$

$$\Delta_5 = \mathbf{C_B A_2} - c_5 = (-2, -1, 0)(3/5, -4/5, 1) + M = M - 2/5$$

$$\Delta_6 = \mathbf{C_B A_2} - c_6 = (-2, -1, 0)(-1/5, -3/5, -1) + M = M - 1/5.$$

Since M is as large as possible, $\Delta_3, \Delta_5, \Delta_6$ are all positive. Consequently, the optimal solution is : $x_1 = 3/5$, $x_2 = 6/5$, max $\mathbf{z} = -12/5$.

Example 14. *Solve the following problem by Big-M-method : Max.* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to :* $x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$, *and* $x_1, x_2, x_3, x_4 \geq 0$.

[IAS (Maths.) 95; Kanpur (B.Sc.) 92; Karala (B.Sc.) 91; Meerut (B.Sc.) 90]

Solution. Since the constraints of the given problem are equations, introduce the artificial variables $a_1 \geq 0$, $a_2 \geq 0$. The problem thus becomes :

Max. $z = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2$, subject to the constraints :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad + a_1 \quad &= 15 \\ 2x_1 + x_2 + 5x_3 \quad + a_2 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \quad &= 10 \\ \text{and } x_1, x_2, x_3, x_4, a_1, a_2 \quad &\geq 0. \end{aligned}$$

Now applying the usual simplex method, the solution is obtained as given in the *Table 5·27*.

Table 5·27 (Example 14)

BASIC VARIABLES	C_B	X_B	X_1 (1)	X_2 (2)	X_3 (3)	X_4 (−1)	A_1 (−M)	A_2 (−M)	MIN RATIO (X_B/X_k)
a_1	−M	15	1	2	3 ↑	0	1	0	15/3
← a_2	−M	20	2	1	← [5]	0	0	1	20/5 ←
x_4	−1	10	1	2	1	1	0	0	10/1
	$z = (-35M - 10)$		$(-3M-2)$	$(-3M-2)$	$(-8M-4)$ ↑	0	0	0 ↓	← Δ_j
← a_1	−M	3	−1/5	[7/5]	0	0	1	×	$\frac{3}{7/5}$ ←
→ x_3	3	4	2/5	1/5	1	0	0	×	$\frac{4}{1/5}$
x_4	−1	6	3/5	9/5	0	1	0	×	$\frac{6}{9/5}$
	$z = (-3M + 6)$		$(M-2)/5$	$-(7M-16)/5$ ↑	0	0	0 ↓	×	← Δ_j
→ x_2	2	15/7	−1/7	1	0	0	×	×	—
x_3	3	25/7	3/7	0	1	0	×	×	25/3
← x_4	−1	15/7	[6/7]	0	0	1	×	×	15/6 ←
	$z = 90/7$		−6/7* ↑	0	0	0 ↓	×	×	← Δ_j
x_2	2	15/6	0	1	0	1/6	×	×	
x_3	3	15/6	0	0	1	3/6	×	×	
→ x_1	1	15/6	1	0	0	7/6	×	×	
	$z = 15$		0	0	0	75/36	×	×	← $\Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been obtained as :

$$x_1 = x_2 = x_3 = \frac{15}{6} = \frac{5}{2}, \text{ max } z = 15 .$$

Example 15. *Use penalty (Big-M) method to maximize* : $z = 3x_1 - x_2$ *subject to the constraints* :

$$2x_1 + x_2 \geq 2,\ x_1 + 3x_2 \leq 3, x_2 \leq 4, \text{ and } x_1, x_2 \geq 0 .$$

Solution. By introducing the surplus variable $x_3 \geq 0$, artificial variable $a_1 \geq 0$, and slack variables $x_4 \geq 0$, $x_5 \geq 0$, the problem becomes : Max. $z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} 2x_1 + x_2 - x_3 \quad + a_1 &= 2 \\ x_1 + 3x_2 \quad + x_4 \quad &= 3 \\ x_2 \quad + x_5 &= 4 \\ x_1, x_2, x_3, x_4, x_5, a_1 &\geq 0 . \end{aligned}$$

In matrix form, $\begin{bmatrix} 2 & 1 & -1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Now the solution is obtained as given in Table 5.28

Table 5·28 [Example 15]

BASIC VARIABLES	C_B	X_B	X_1 (c_j = 3)	X_2 (c_j = −1)	X_3 (c_j = 0)	X_4 (c_j = 0)	X_5 (c_j = 0)	A_1 (c_j = −M)	MIN. RATIO (X_B/X_k)
$\leftarrow a_1$	$-M$	2	[2]	1	1	0	0	1	$2/2 \leftarrow$
x_4	0	3	1	3	0	1	0	0	3/1
x_5	0	4	0	1	0	0	1	0	—
	$z = -2M$		$(-2M-3)\uparrow$	$-M+1$	M	0	0	$0\downarrow$	
$\rightarrow x_1$	3	1	1	1/2	$-1/2$	0	0	×	—
$\leftarrow x_4$	0	2	0	5/2	[1/2]	1	0	×	$2/\frac{1}{2} \leftarrow$
x_5	0	4	0	1	0	0	1	×	—
	$z = 3$		0	5/2	$(-3/2)\uparrow$	$0\downarrow$	0	×	$\leftarrow \Delta_j$
x_1	3	3	1	3	0	1	0	×	
$\rightarrow x_3$	0	4	0	5	1	2	0	×	
x_5	0	4	0	1	0	0	1	×	
	$z = 9$		0	10	0	3	0	×	$\leftarrow \Delta_j \geq 0$

Thus the optimum solution is obtained as : $x_1 = 3$, $x_2 = 0$, max. $z = 9$.

Example 16. ***(Unrestricted Variables)***

(a) Maximize $z = 8x_2$, *subject to the constraints :* $x_1 - x_2 \geq 0, 2x_1 + 3x_2 \leq -6$ *and* x_1 , x_2 *are unrestricted.* **[Meerut (Maths) 93]**

(b) Solve the LPP : Max $z = 4x_1 + 6x_2$, *subject to :* $x_1 - 2x_2 \geq -4$, $2x_1 + 44x_2 \geq 12$, $x_1 + 3x_2 \geq 9$ *and* x_1, x_2 *are unrestricted.* **[JNTU (MCA) 2004; Meerut 96, 93]**

Solution. (a) In this problem, the variables x_1 and x_2 are unrestricted in sign, *i.e.* x_1 and x_2 may be + ive, –ive or zero. But, the simplex method can be used only when the variables are non-negative (≥ 0) . This difficulty can be immediately removed by using the transformation :

$$x_1 = x_1' - x_1'' \quad \text{and} \quad x_2 = x_2' - x_2'' \text{ such that } x_1' \geq 0 , x_1'' \geq 0 , x_2' \geq 0 , x_2'' \geq 0 .$$

Therefore, the given problem becomes : *maximize* $z = 8x_2' - 8x_2''$, *subject to the constraints :*

$$(x_1' - x_1'') - (x_2' - x_2'') \geq 0$$
$$-2(x_1' - x_1'') - 3(x_2' - x_2'') \geq 6$$
$$x_1' , x_1'' , x_2' , x_2'' \geq 0 .$$

Now introducing the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the given problem becomes : Max. $z = 0x_1' + 0x_1'' + 8x_2' - 8x_2'' + 0x_3 + 0x_4 - Ma_1 - Ma_2$, subject to :

$$x_1' - x_1'' - x_2' + x_2'' - x_3 + a_1 = 0$$
$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - x_4 + a_2 = 6$$
$$x_1' , x_1'' , x_2' , x_2'' , x_3 , x_4 , a_1 , a_2 \geq 0 .$$

Table 5.29

BASIC VAR.	C_B	X_B	X_1' (c_j = 0)	X_1'' (c_j = 0)	X_2' (c_j = 8)	X_2'' (c_j = −8)	X_3 (c_j = 0)	X_4 (c_j = 0)	A_1 (c_j = −M)	A_2 (c_j = −M)	MIN RATIO (X_B/X_k)
$\leftarrow a_1$	$-M$	0	1	-1	-1	[1]	-1	0	1	0	$0 \leftarrow$
a_2	$-M$	6	-2	2	-3	3	0	-1	0	1	6/3
	$z = -6M$		M	$-M$	$(4M-8)$	$(-4M+8)\uparrow$	M	M	$0\downarrow$	1	$\leftarrow \Delta_j$
$\rightarrow x_2'$	-8	0	1	-1	-1	1	-1	0	×	0	—
$\leftarrow a_2$	$-M$	6	-5	[5]	0	0	3	-1	×	1	$6/5 \leftarrow$
	$z = -6M$		$(5M-8)$	$(-5M+8)\uparrow$	0	0	$(-3M+8)$	M	×	$0\downarrow$	$\leftarrow \Delta_j$
x_2''	-8	6/5	0	0	-1	1	$-2/5$	1/5	×	×	
$\rightarrow x_1''$	0	6/5	-1	1	0	0	3/5	$-1/5$	×	×	
	$z = -48/5$		0	0	0	0	16/5	8/5	×	×	$\leftarrow \Delta_j \geq 0$

Remember that the coefficients of slack or surplus variables in the objective function are always zero and the coefficient of artificial variables is taken a *largest negative quantity* $-M$ *where* $M > 0$.

Applying the simplex method in the usual manner, the solution is obtained as given in *Table 5·29*.

Since all $\Delta_j \geq 0$, an optimum solution is obtained as : $x_1' = 0$, $x_1'' = 6/5$, $x_2' = 0$, $x_2'' = 6/5$.

Since $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$, transforming the solution to original variables, we get

$$x_1 = 0 - 6/5 = -6/5,\ x_2 = 0 - 6/5 = -6/5,\ \text{max. } z = -48/5.$$

(b) Solve as (a).

Note. Whenever the range of a variable is not given in the problem, it should be understood that such variable is unrestricted in sign.

Example 17. (Imp.) *Maximize* $z = 4x_1 + 5x_2 - 3x_3 + 50$, *subject to the constraints :*

$$x_1 + x_2 + x_3 = 10 \quad ...(i)$$
$$x_1 - x_2 \geq 1 \quad ....(ii)$$
$$2x_1 + 3x_2 + x_3 \leq 40 \quad ...(iii)$$
$$x_1, x_2, x_3 \geq 0.$$

[Meerut (Maths.) 97 P]

Solution. *If any constant is included in the objective function (like 50 here) it should be deleted in the beginning and finally adjusted in optimum value of z and, if there is an equality in the constraints, then one variable can be eliminated from the inequalities with* $\leq$ or $\geq$ *sign. (Note)*

Subtracting *(i)* from *(iii)* with a view to eliminate x_3 from (iii) and retaining x_3 in (i) to work as a slack variable, the restrictions are modified as follows :

$$x_1 + x_2 + x_3 = 10, x_1 - x_2 \geq 1, x_1 + 2x_2 \leq 30, \text{ and } x_1, x_2, x_3 \geq 0.$$

Now introducing the slack, surplus and artificial variables, the problem becomes :

Max. $z = 4x_1 + 5x_2 - 3x_3 + 0x_4 - Ma_1 + 0x_5$, subject to the constraints :

$$x_1 + x_2 + x_3 = 10$$
$$x_1 - x_2 - x_4 + a_1 = 1$$
$$x_1 + 2x_2 + x_5 = 30$$
$$x_1, x_2, x_3, x_4, x_5, a_1 \geq 0.$$

Applying the usual simplex method, the solution is obtained as given in Table 5·30.

Table 5·30 [Example 17]

		$c_j \rightarrow$	4	5	-3	0	$-M$	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	X_5	MIN. RATIO (X_B/X_k)
x_3	-3	10	1	1	1	0	0	0	10/1
$\leftarrow a_1$	$-M$	1	$\leftarrow$ [1]	1	0	-1	1	0	$1/1 \leftarrow$
x_5	0	30	1	2	0	0	0	1	30/1
	$z = -30 - M$		$-7-M$ $\uparrow$	$-8+M$	0	M	0 $\downarrow$	0	$\leftarrow \Delta_j$
$\leftarrow x_3$	-3	9	0	[2]	1	1	$\times$	0	$9/2 \leftarrow$
$\rightarrow x_1$	4	1	1	-1	0	-1	$\times$	0	—
x_5	0	29	0	3	0	1	$\times$	1	29/3
	$z = 9/2$		0	-15^* $\uparrow$	0 $\downarrow$	-7	$\times$	0	$\leftarrow \Delta_j$
$\rightarrow x_2$	5	9/2	0	1	1/2	1/2	$\times$	0	
x_1	4	11/2	1	0	1/2	$-1/2$	$\times$	0	
x_5	0	31/2	0	0	$-3/2$	$-1/2$	$\times$	1	
	$z = 89/2$		0	0	15/2	1/2	$\times$	0	$\leftarrow \Delta_j \geq 0$

Hence the solution is : $x_1 = 11/2$, $x_2 = 9/12$, max. $z = 89/2 + 50 = 189/2$.

Example 18. *Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A per gram and 12 units of vitamin B and costs 20 paise per gram. The daily minimum requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the minimum cost of product mix by simplex method.* **[Bharthidasan B.Sc. (Math.) 90]**

Solution. Let x_1 grams of food X and x_2 grams of food Y be purchased. Then the problem can be formulated as : Minimize $z = 12x_1 + 20x_2$, subject to the constraints : $6x_1 + 8x_2 \geq 100$, $7x_1 + 12x_2 \geq 120$, and $x_1, x_2 \geq 0$.

Introducing the surplus variables $x_3 \geq 0, x_4 \geq 0$ and artificial variables $a_1 \geq 0, a_2 \geq 0$, the constraints become :

$$6x_1 + 8x_2 - x_3 \quad + a_1 \quad = 100$$
$$7x_1 + 12x_2 \quad - x_4 \quad + a_2 = 120.$$

Objective function becomes :

$$\text{Max. } z' = -12x_1 - 20x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2, \text{ where } z' = -z.$$

Now proceeding by usual simplex method, the solution is obtained as given in Table 5.31.

Table 5·31 [Example 18]

	$c_j \rightarrow$		-12	-20	0	0	$-M$	$-M$	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{A_1}$	$\mathbf{A_2}$	MIN RATIO $(\mathbf{X_B/X_k})$
a_1	$-M$	100	6	8 ↑	-1	0	1	0	100/8
$\leftarrow a_2$	$-M$	120	7	← [12]	0	1	0	1	120/12 ←
	$z' = -220M$		$(-13M+12)$	$(-20M+20)$ ↑	M	M	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	$-M$	20	[4/3]	0	-1	2/3	1	×	60/4 ←
$\rightarrow x_2$	-20	10	7/12	1	0	$-11/2$	0	×	120/7
	$z' = -20M - 200$		$-(4M-1)/3$ ↑	0	M	$(-2M+5)/3$	0 ↓	×	$\leftarrow \Delta_j$
$\rightarrow x_1$	-12	15	1	0	$-3/4$	1/2	×	×	
x_2	-20	5/4	0	1	7/16	$-3/4$	×	×	
	$z' = -205$		0	0	1/4	9	×	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal solution is attained. Hence the optimal solution is :

$$x_1 = 15, x_2 = 5/4, \text{ max } z = -(-205) = 205.$$

Hence 15 grams of food X and 5/4 grams of food Y should be the required product-mix with minimum cost of Rs. 205.

5·6. DISADVANTAGES OF BIG-M-METHOD OVER TWO-PHASE METHOD

1. Although big-M method can always be used to check the existence of a feasbile solution, it may be computationally inconvenient because of the manipulation of the constant M. On the other hand, two-phase method eliminates the constant M from calculations.
2. Another difficulty arises specially when the problem is to be solved on a digital computer. To use a digital computer, M must be assigned some numerical value which is much larger than the values $c_1, c_2, \ldots$, in the objective function. But, a computer has only a fixed number of digits.

Q. 1. What is an artificial variable and why it is necessary to introduce it ? Describe the two phase process of solving an L.P.P. by simplex method. **[Delhi B.Sc. (Maths.) 90]**

2. Why is an artificial vector which leaves the basis once never considered again for re-entry into the basis ? **[Delhi B.Sc. (Math.) 91]**

3. In the two-phase method explain when phase 1 terminates.

4. Optimality criteria being satisfied; state what is indicated by each of the following :
(i) One or more artificial vectors are in the basis at zero level.
(ii) One or more artificial vectors are in the basis at positive level. **[Delhi B.Sc. (Math.) 91]**

EXAMINATION PROBLEMS

Solve the following LP problems using Charne's Big-M (Penalty) method:

1. Min. $z = 2x_1 + 9x_2 + x_3$, subject to the constraints : $x_1 + 4x_2 + 2x_3 \geq 5$, $3x_1 + x_2 + 2x_3 \geq 4$, $x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 0$, $x_2 = 0$, $x_3 = 5/2$, min. $z = 5/2$].

2. Max. $R = 5x - 2y - z$, subject to the constraints : $2x + 2y - z \geq 2$, $3x - 4y \geq 3$, $y + 3z \geq 5$, $x, y, z \geq 0$.
[Ans. $x = 13/9$, $y = 1/3$, $z = 14/9$, max. $R = 5$]

3. Min. $z = x_1 + x_2 + 3x_3$, subject to $3x_1 + 2x_2 + x_3 \leq 3$, $2x_1 + x_2 + 2x_3 \geq 3$ $x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 3/4$, $x_2 = 0$, $x_3 = 3/4$, min. $z = 3$]

4. Min. $z = 4x_1 + 8x_2 + 3x_3$, subject to $x_1 + x_2 \geq 2$, $2x_1 + x_3 \geq 5$ and $x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$, min. $z = 10$]

5. Maximize $z = 2x_1 + x_2 + 3x_3$, subject to $x_1 + x_2 + 2x_3 \leq 5$, $2x_1 + 3x_2 + 4x_3 = 12$, and $x_1, x_2, x_3 \geq 0$.
What is the maximum number of basic solutions to the L.P. problem ?
[Ans. $x_1 = 3$, $x_2 = 2$, $x_3 = 0$, max. $z = 8$]

6. Min. $P = 5x + 4y + 4z$, subject to $x + y + z = 100$, $x \geq 20$, $y \geq 30$, $z \leq 40$ and $x, y, z \geq 0$,
[JNTU (B. Tech.) 2003]

7. Maximize $z = 3x_1 + 2{\cdot}5x_2$, subject to the constraints : $2x_1 + 4x_2 \geq 40$, $3x_1 + 2x_2 \geq 50$ $x_1, x_2 \geq 0$
[Hint. First constraint can be divided by the common factor 2]
[Ans. Unbounded solution]

8. Min. $z = 0{\cdot}6x_1 + x_2$, subject to the constraints : $10x_1 + 4x_2 \geq 20$, $5x_1 + 5x_2 \geq 20$, and $x_1, x_2 \geq 0$.
[Hint. First divide the constraints by the common factors 2,5,2 respectively]
[Ans. $x_1 = 4$, $x_2 = 0$, min. $z = 12/5$]

9. Min. (cost) $z = 2y_1 + 3y_2$, subject to the constraints : $y_1 + y_2 \geq 5$, $y_1 + 2y_2 \geq 6$, $y_1 \geq 0$, $y_2 \geq 0$
[Ans. $y_1 = 4$, $y_2 = 1$ min. $z = 11$]

10. Max. $z = 3x_1 + 2x_2$, subject to the constraints : $2x_1 + x_2 \leq 2$, $3x_1 + 4x_2 \geq 12$, and $x_1, x_2 \geq 0$.
[Ans. Pseudo-optimum basic feasible solution exist.]

11. Min. $z = 4x_1 + 2x_2$, subject to the constraints : $3x_1 + x_2 \geq 27$, $x_1 + x_2 \geq 21$, and $x_1, x_2 \geq 0$.
[Ans. $x_1 = 3$, $x_2 = 18$, min $z = 48$]

12. Max. $z = 2x_1 + 3x_2 - 5x_3$, subject to the constraints : $x_1 + x_2 + x_3 = 7$, $2x_1 + 5x_2 + x_3 \geq 10$, and $x_1, x_2, x_3 \geq 0$.
[Hint. Since first constraint is an equation, we can subtract this equation from second constraint n order to reduce the number of artificial variables]
[Ans. $x_1 = 45/7$, $x_2 = 4/7$, $x_3 = 0$, max. $z = 102/7$]

13. Min. $z = 3x_1 + 2x_2 + x_3$, subject to : $2x_1 + 5x_2 + x_3 = 12$, $3x_1 + 4x_2 = 11$ x_1 is unrestricted, $x_2 \geq 0$, $x_3 \geq 0$.
[Ans. $x_1 = 11/3$, $x_2 = 0$, $x_3 = 14/3$, max $z = 47/3$.]

14. Min. $z = 5x_1 + 6x_2$, subject to $2x_1 + 5x_2 \geq 1500$, $3x_1 + x_2 \geq 1200$, and $x_1, x_2 \geq 0$. Verify the result graphically.
[Kanpur (B.Sc.) 95, 91]

15. A cabinet manufacturer produces wood cabinets for T.V., sets, stereo systems and radios, each of which must be assembled and crated. Each T.V. cabinet requires 3 hrs. to assemble, 5 hrs. to decorate and 1/10 hr. to crate and returns a profit of Rs. 10. Each stereo cabinet requires 10 hrs. to assemble 8 hrs. to decorate and 3/5 hr. to crate and returns a profit Rs. 25. Each radio cabinet requires 1 hr. to assemble, 1 hr. to decorate and 1/10 hr. to, crate and returns a profit of Rs. 3. The manufacturer has the maximum of 30,0[illegible] 40,000 and 120 hrs. available for assembling, decorating and crating respectively.
(i) Formulate the above problem as a LPP.
(ii) Use simplex method to find how many units of each product should be manufactured to maximize profit.
(iii) Does the problem have unique solution. **[Delhi (MBA) Nov. 98]**
[Ans. (i) Max. $z = 10x_1 + 25x_2 + 3x_3$, s.t.
$$3x_1 + 10x_2 + x_3 \leq 30{,}000,\ 5x_1 + 8x_2 + x_3 \leq 40{,}000,$$
$$\frac{1}{10}x_1 + \frac{3}{5}x_2 + \frac{1}{10}x_3 \leq 120,\ x_1, x_2, x_3 \geq 0$$
(ii) $x_1 = 1{,}200$, $x_2 = x_3 = 0$ with max. profit $z = 12{,}000$.
(iii) No.]

16. Product *A* offers a profit of Rs. 25 per unit and product *B* yields a profit of Rs. 40 per unit. To manufacture the products-leather, wood and glue are required in the amount shown below :

Resources Required for one unit

Product	Leather	Woods (in sq. units)	Glue (in litres)
A	0·50	4	0·2
B	0·25	7	0·2

Available resources include 2,200 kgs of leather, 28,000 square metres of wood and 1,400 litres of glue :
(i) State the objective function and constraints in mathematical form.
(ii) Find the optimum situation.
(iii) Which resources are fully consumed ? How much of each resource remains unused ?
(iv) What are the shadow prices of resources ? **[C.S. (Final) June 97]**
[Hint.(i) Max. $z = 25x_1 + 40x_2$, s.t.
$$0{\cdot}50x_1 + 0{\cdot}25x_2 \geq 2{,}200,\ 4x_1 + 7x_2 \leq 28{,}000,$$
$$0{\cdot}20x_1 + 0{\cdot}20x_2 \leq 1{,}400,\ x_1, x_2 \geq 0.$$
(ii) $x_1 = 3{,}360$, $x_2 = 2{,}080$ with max. $z =$ Rs. 1,67,200

(iii) used and unused reosurces.

Resources	Used	Unused	Total
Leather	(0·50) (3360) + (0·25) (2080) = 2200 kg.	0 kg.	2,200 kg.
Wood	28,000 sq. mt.	0 sq. mt.	28,000 sq. mt.
Glue	1080 litres	312 litres	1,400 litres

16. Two products A and B are processed on 3 machine, M_1, M_2 and M_3. The processing times per unit, machine availability and profit per unit are

Machine	Processing	Time (hrs.)	Availability (hrs.)
M_1	2	3	1,500
M_2	3	2	1,500
M_3	1	1	100
Profit (Rs.)	10	12	

Any unutilized time on machine M_3 can be given on rental basis to others at an hourly rated of Rs. 1·50. Solve the problem by simplex method to determine the maximum profit. **[IGONU (MBA) Dec. 98]**

[**Hint.** Since any amount of unused time on M_3 can be rented out at a rate of Rs. 1·50 per hour, the total rent will be 1·5 [1000 – (x_1 + x_2)]. Thus the total profit is equal to $10x_1 + 12x_2 + 1{,}500 - 1{\cdot}5x_1 - 1{\cdot}5x_2$. Thus formulation of LPP is :

Max. $z = 8{\cdot}5x_1 + 10{\cdot}5x_2 + 1{,}500$, s.t. $2x_1 + 3x_2 \le 1{,}500$, $3x_1 + 2x_2 \le 1{,}500$, $x_1 + x_2 \le 1{,}000$; $x_1, x_2 \ge 0$.

[**Ans.** 300 units of A and B both with max. profit of Rs. 7,200]

17. A factory manufactures three products which are processed through three different production stages. The time required to manufacture, one unit of each of the three products and their daily capacity of the stages are given in the following table :

Stage	Time per unit in minutes			Stage capacity (in minutes)
	Product 1	Product 2	Product 3	
1	1	1	1	430
2	3	—	2	460
3	1	4	—	420
Profit per unit				—

(i) Set the data in simplex table.
(ii) Find the table for optimum solution.
(iii) State from the table-min. profit, production pattern and surplus capacity at any stage.
(iv) What is the meaning of shadow price ? Where is it shown in the table ? Explain it in respec ɹ resource of stages having shadow price.
(v) How many units of other resources will be required so as to completely utilise the surplus resource ?
[Osmania (MBA) Feb. 97]

18. Ashok Chemicals Co. manufactures two chemicals A and B which are sold to the manufacturers of soaps and detergents. On the basis of the next month's demand, the management has decided that the total production for chemicals A and B should be at least 350 kilograms. Moreover, a major customer's order for 125 kgs. of product A must also be supplied. Product A requires 2 hours of processing time per kg. and product B requires one hour of processing time per kg. For the coming month, 600 hours of processing time are available. The company wants to meet the above requirements at a minimum total production cost. The production costs are Rs. 2/– per kg. for product A and Rs. 3/– per kg for product B.

Ashok Chemicals Co. wants to determine its optimum productwise and the total minimum cost relevant thereto.
(i) Formulate the above as a linear programming problem.
(ii) Solve the problem with the simplex method.
(iii) Does the problem have multiple optimum solutions ? **[Delhi (M. Com.) 98]**

19. A firm manufacturing office furniture provides you the following information regarding resource consumption and availability and profit contribution :

Resources	Usage per unit			Daily availability
	Tables	Chairs	Bookcases	
Timber (cu. ft)	8	4	3	640
Assembly department (man-hours)	4	6	2	540
Finishing department (man-hours)	1	1	1	100
Profit contribution per unit (Rs.)	30	20	12	
Minimum production requirement	0	50	0	

The firm wants to determine its optimal product mix.

(i) Formulate the linear programming problem with the help of the above data.
(ii) Solve the problem with the Simplex Method and find the optimal product mix and the total maximum profit contribution.
(iii) Identify the shadow prices of the resources.
(iv) What other information can be obtained from the optimal solution of the problem ? **[Delhi (M. Com.) 97]**

20. Use penalty (Big M) method to solve the following LP problem :
Min. $z = 5x_1 + 3x_2$, s.t. $2x_1 + 4x_2 \leq 12$, $2x_1 + 2x_2 = 10$, $5x_1 - 2x_2 \geq 10$, and $x_1, x_2 \geq 0$ **[IPM (PGDBM) 2000]**

Problem of Degeneracy (Tie for Minimum Ratio)

5.7. WHAT IS DEGENERACY PROBLEMS ?

At the stage of improving the solution during simplex procedure, minimum ratio $\mathbf{X_B/X_k}$ $(\mathbf{X_k > 0})$ is determined in the last column of simplex table to find the key row (*i.e.*, a row containing the *key element*). But, sometimes this ratio may not be unique, *i.e.*, the key element (hence the variable to leave the basis) is not uniquely determined or at the very first iteration, the value of one or more basic variables in the $\mathbf{X_B}$ column become equal to zero, this causes the problem of degeneracy.

However, if the minimum ratio is zero for two or more basic variables, degeneracy may result the simplex routine to cycle indefinitely. That is, the solution which we have obtained in one iteration may repeat again after few iterations and therefore no optimum solution may be obtained under such circumstances. Fortunately, such phenomenon very rarely occurs in practical problems.

5.7–1. Method to Resolve Degeneracy (Tie)

The following systematic procedure can be utilised to avoid cycling due to degeneracy in L.P. problems.

Step 1. First pick up the rows for which the min. non-negative ratio is same (tied). To be definite, suppose such rows are first, third, etc., for example.

Step 2. Now rearrange the columns of the usual simplex table so that the columns forming the original unit matrix come first in proper order.

Step 3. Then find the minimum of the ratio :

$$\left[\frac{\textbf{elements of first column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique. That is, for the rows *first, third,* etc. as picked up in step 1. (*key column* is that one for which Δ_j is minimum).

(i) If this minimum is attained for third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is also not unique, then go to next step.

Step 4. Now compute the minimum of the ratio :

$$\left[\frac{\textbf{elements of second column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique in *Step* 3.

(i) If this min. ratio is unique for the first row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is still not unique then go to next step.

Step 5. Next compute the *minimum* of the ratio :

$$\left[\frac{\textbf{elements of third column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique in *Step* 4.

(i) If this min. ratio is unique for the third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this min. is still not unique, then go on repeating the above outlined procedure till the unique min. ratio is obtained to resolve the degeneracy. After the resolution of this tie, simplex method is applied to obtain the optimum solution. Following example will make the procedure clear.

Q. 1. What do you understand by degeneracy ? Discuss a method to resolve degeneracy in a LPP.
2. Explain the concept of degeneracy in simplex method. **[VTU 2003]**

Example 19. *Maximize* $\mathbf{z} = 3x_1 + 9x_2$, *subject to the constraints* :

$$x_1 + 4x_2 \le 8,\ x_1 + 2x_2 \le 4, \text{ and } x_1, x_2 \ge 0.$$

Solution. Introducing the slack variables $s_1 \ge 0$ and $s_2 \ge 0$, the problem becomes :

$$\text{Max. } z = 3x_1 + 9x_2 + 0s_1 + 0s_2$$

subject to the constraints :

$$x_1 + 4x_2 + s_1 = 8$$
$$x_1 + 2x_2 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Table 5.32. Starting Simplex Table

		$c_j \rightarrow$	3	9	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1	S_2	MIN. RATIO (X_B/X_k)
s_1	0	8	1	4	1	0	$8/4 = 2$ Tie
s_2	0	4	1	2	0	1	$4/2 = 2$
	$z = 0$		-3	-9	0	0	$\leftarrow \Delta_j$

Since min. ratio 2 in the last column of above table is not unique, both the slack variables s_1 and s_2 may leave the basis. This is an indication for the existence of degeneracy in the given LP problem. So we apply the above outlined procedure to resolve degeneracy (tie).

First arrange the columns $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{S}_1$ and $\mathbf{S}_2$ in such a way that the initial identity (basis) matrix appears first. Thus the initial simplex table becomes :

Table 5.33

		$c_j \rightarrow$	0	0	3	9	
BASIC VARIABLES	C_B	X_B	S_1	S_2	X_1	X_2	MIN RATIO (S_1/X_2)
s_1	0	8	1	0	1	4	1/4
$\leftarrow s_2$	0	4	0	1	1	$\leftarrow$ [2]	0/2 $\leftarrow$
	$z = 0$		0	0 ↓	-3	-9 ↑	$\leftarrow \Delta_j \ge 0$

Now using the *step 3* of the procedure for resolving degeneracy, we find

$$\min \left[\frac{\textbf{elements of first column } (\mathbf{S}_1)}{\textbf{corres. elements of } \textbf{\textit{key column}} \ (\mathbf{X}_2)} \right] = \min \left[\tfrac{1}{4}, \tfrac{0}{2} \right] = 0$$

which occurs for the second row. Hence $\mathbf{S}_2$ must leave the basis, and the key element is 2 as shown above.

First Iteration. By usual matrix transformation introduce $\mathbf{X}_2$ and leave $\mathbf{S}_2$.

Table 5.34. First Improvement Table

		$C_j \rightarrow$	0	0	3	9	
BASIC VARIABLES	C_B	X_B	S_1	S_2	$\mathbf{X}_1$	$\mathbf{X}_2$	MIN RATIO
s_1	0	0	1	-2	-1	0	
$\rightarrow x_2$	9	2	0	1/2	1/2	1	
	$z = 18$		0	9/2	3/2	0	$\leftarrow \Delta_j \ge 0$

Since all $\Delta_j \ge 0$, an optimal solution has been reached. Hence the optimum basic feasible solution is : $x_1 = 0, x_2 = 2$, max. $\mathbf{z} = 18$.

Example 20. *Max.* $z = 2x_1 + x_2$, *subject to* $4x_1 + 3x_2 \le 12$, $4x_1 + x_2 \le 8$, $4x_1 - x_2 \le 8$, *and* $x_1, x_2 \ge 0$.

Solution. Introducing the slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$, and proceeding in the usual manner, the starting simplex table is given below :

Table 5.35

BASIC VARIABLES	C_B	$c_j \rightarrow$ / X_B	2 / X_1	1 / X_2	0 / S_1	0 / S_2	0 / S_3	MIN. RATIO (X_B/X_k)
s_1	0	12	4	3	1	0	0	12/4
s_2	0	8	4	1	0	1	0	8/4
s_3	0	8	4	−1	0	0	1	8/4
		$z = 0$	−2	−1	0	0	0	$\leftarrow \Delta_j$

Since min. ratio in the last column of above table is 2 which is same for *second* and *third* rows. This is an indication of *degeneracy*. So arrange the columns in such a way that the initial identity (basis) matrix comes first. Then starting simplex table becomes.

Table 5.36

BASIC VARIABLES	C_B	X_B	S_1	S_2	S_3	X_1	X_2	MIN (S_1/X_1)	MIN (S_2/X_1)
s_1	0	12	1	0	0	4	3	—	—
s_2	0	8	0	1	0	4	1	0/4	1/4
s_3	0	8	0	0	1	← [4]	1	0/4	0/4 ←
		$z = 0$	0	0	0 ↓	−2 ↑	−1	$\leftarrow \Delta_j$	

Using the procedure of degeneracy, compute

$$\left[\frac{\text{elements of first column } (S_1) \text{ of unit matrix}}{\text{corres. elements of key column } (X_1)}\right]$$

only for *second* and *third* rows. Therefore, min $[-, \frac{0}{4}, \frac{0}{4}]$ which is not unique.

So again compute

$$\min\left[\frac{\text{element of second column } (S_2) \text{ of unit matrix}}{\text{corres. element of key column } (X_1)}\right]$$

only for second and third rows. Therefore, min $[-, \frac{1}{4}, \frac{0}{4}] = 0$ which occurs corresponding to the third row. Hence the key elemen[illegible] 4.

Now improve the simplex **Table 5.36** in the usual manner to get Table 5.37.

Table 5.37

BASIC VARIABLES	C_B	$c_j \rightarrow$ / X_B	0 / S_1	0 / S_2	0 / S_3	2 / X_1	1 / X_2	MIN. (X_B/X_k)
s_1	0	4	1	0	−1	0	4	4/4
s_2	0	0	0	1	−1	0	← [2]	0/2 ←
x_1	2	2	0	0	1/4	1	−1/4	—
		$z = 4$	0	0 ↓	1/2	0	−3/2 ↑	$\leftarrow \Delta_j$
s_1	0	4	1	−2	[1]	0	0	4/1 ←
x_2	1	0	0	1/2	−1/2	0	1	—
x_1	2	2	0	1/8	1/8	1	0	$2/\frac{1}{8}$
		$z = 4$	0 ↓	3/4	−1/4 ↑	0	0	$\leftarrow \Delta_j$
s_3	0	4	1	−2	1	0	0	
x_2	1	2	1/2	−1/2	0	0	1	
x_1	2	3/2	−1/8	3/8	0	1	0	
		$z = 5$	1/4	1/4	0	0	0	$\leftarrow \Delta_j \ge 0$

Since all $\Delta_j \ge 0$, an optimum solution is obtained as : $x_1 = 3/2$, $x_2 = 2$, max $z = 5$.

Example 21. *Max.* $z = 5x_1 - 2x_2 + 3x_3$, *subject to* $2x_1 + 2x_2 - x_3 \geq 2$, $3x_1 - 4x_2 \leq 3$, $x_2 + 3x_3 \leq 5$, *and* $x_1, x_2, x_3 \geq 0$. **[Kanpur 96]**

Solution. Introducing the surplus variable $s_1 \geq 0$, slack variables $s_2 \geq 0, s_3 \geq 0$ and an artificial variable $a_1 \geq 0$, the constraints of the problem become :

$$2x_1 + 2x_2 - x_3 - s_1 \quad + a_1 = 2$$
$$3x_1 - 4x_2 \quad + s_2 \quad = 3$$
$$x_2 + 3x_3 \quad + s_3 \quad = 5.$$

and using big-M technique objective function becomes :

$$\text{Max. } z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - Ma_1.$$

In the usual manner, the starting simplex table is obtained as below :

Table 5.38

		$c_j \rightarrow$	5	−2	3	0	0	0	−M	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	A_1	MIN. RATIO (X_B/X_k)
$\leftarrow a_1$	−M	2	←[2]	2	−1	−1	0	0	1	2/2←
s_2	0	3	3	−4	0	0	1	0	0	3/3
s_3	0	5	0	1	3	0	0	1	0	–
	z = −2M		−2M−5 ↑	−2M+2	M−3	M	0	0	0 ↓	$\leftarrow \Delta_j$

Net evaluations Δ_j are computed by the formula $\Delta_j = C_B X_j - c_j$ in the usual manner. Since Δ_1 is the most negative, X_1 enters the basis. Further, since the min. ratio in the last column of above table is 1 for both the first and second rows, therefore either A_1 or S_2 tends to leave the basis. This is an indication of the existence of degeneracy. But, A_1 being an artificial vector will be prefered to leave the basis. *Note that there is no need to apply the procedure for resolving degeneracy under such circumstances.*

Continuing the simplex routine, the computations are presented in the following tabular form.

Table 5.39

		$c_j \rightarrow$	5	−2	3	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	MIN. RATIO (X_B/X_k)
$\rightarrow x_1$	5	1	1	1	−1/2	−1/2	0	0	–
$\leftarrow s_2$	0	0	0	−7	←[3/2]	−3/2	1	0	$0/\frac{3}{2}$←
s_3	0	5	0	1	3	0	0	1	5/3
	z = 5		0	7	−11/2 ↑	−5/2	0 ↓	0	$\leftarrow \Delta_j$
x_1	5	1	1	−4/3	0	0	1/3	0	–
$\rightarrow x_3$	3	0	0	−14/3	1	1	2/3	0	–
$\leftarrow s_3$	0	5	0	[15]	0	−3	−2	1	5/15←
	z = 5		0	−56/3 ↑	0	3	11/3	0 ↓	$\leftarrow \Delta_j$
x_1	5	13/9	1	0	0	−4/15	7/45	4/45	–
$\leftarrow x_3$	3	14/9	0	0	1	[1/15]	2/45	14/45	70/3←
$\rightarrow x_2$	−2	1/3	0	1	0	−1/5	−2/15	1/15	–
	z = 101/9		0	0	0 ↓	−11/15 ↑	53/45	56/45	$\leftarrow \Delta_j$
x_1	5	23/3	1	0	4	0	1/3	4/3	
$\rightarrow s_1$	0	70/3	0	0	15	1	2/3	14/3	
x_2	−2	5	0	1	3	0	0	1	
	z = 85/3		0	0	11	0	5/3	14/3	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, optimum solution is : $x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$, max. $z = 85/3$.

Q. 1. What is degeneracy ? Discuss a method to resolve degeneracy in L.P. problems.

2. Explain what is meant by degeneracy and cycling in linear programming. How their effects overcome ?

[Meerut (L.P.) 90]

EXAMINATION PROBLEMS

Solve the following LP problems :

1. Max. $z = 5x_1 + 3x_2$ subject to
$x_1 + x_2 \leq 2$
$5x_1 + 2x_2 \leq 10$
$3x_1 + 8x_2 \leq 12$
$x_1, x_2, \geq 0$.
[Ans. $x_1 = 2$, $x_2 = 0$, $z = 10$]

2. Max. $R = 22x + 30y + 25z$ subject to
$2x + 2y \leq 100$
$2x + y + z \leq 100$
$x + 2y + 2z \leq 100$
$x, y, z, \geq 0$.
[Ans. $x = 100/3$, $y = 50/3$, $z = 50/3$, $R = 1650$]

3. Max. $z = 2x_1 + 3x_2 + 10x_3$ subject to
$x_1 + 2x_3 = 0$
$x_2 + x_3 = 1$
$x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 0$, $x_2 = 1$, $x_3 = 0$ and max. $z = 3$]

4. Max. $z = 3x_1 + 5x_2$ subject to the constraints
$x_1 + x_3 = 4$, $x_2 + x_4 = 6$,
$3x_1 + 2x_2 + x_5 = 12$, and
$x_1, x_2, x_3, x_4, x_5 \geq 0$
Does the degeneracy occur in this problem ?
[Ans. $x_1 = 0$, $x_2 = 6$, $x_3 = 4$, $x_4 = 0$, $x_5 = 0$, $z^* = 30$. Yes, degeneracy occurs.]

5. Max. $z = 2x_1 + x_2$ subject to the constraints
$x_1 + 2x_2 \leq 10$, $x_1 + x_2 \leq 6$,
$x_1 - x_2 \leq 2$, $x_1 + 2x_2 \leq 1$,
$2x_1 - 3x_2 \leq 1$, and $x_1, x_2 \geq 0$.
[Ans. $x_1 = 5/7$, $x_2 = 1/7$ max. $z = 11/7$]

6. Max. $z = 3/4\, x_1 - 150\, x_2 + 1/50\, x_3 - 6x_4$, subject to the constraints
$1/4\, x_1 - 60\, x_2 - 1/26\, x_3 + 9\, x_4 \leq 0$,
$1/2\, x_1 - 90\, x_2 - 1/50\, x_3 + 3x_4 \leq 0$,
$x_3 \leq 1$ and $x_1, x_2, x_3, x_4 \geq 0$.
[Ans. $x_1 = 1/25$, $x_2 = 0$, $x_3 = 1$ and $x_4 = 0$, max. $z = 1/20$]

7. Max. $z = 2x_1 + 3x_2 + 10x_3$, subject to
$x_1 + 2x_3 = 1$, $x_2 + x_3 = 1$,
and $x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 0$, $x_2 = 1/2$, $x_3 = 1/2$, max. $z = 13/2$]

8. Min. $z = -3/4\, x_1 + 20x_2 - 1/2\, x_3 + 6x_4$, subject to
$1/4\, x_1 - 8x_2 - x_3 + 9x_4 \leq 0$, $1/4\, x_1 - 12x_2 - 1/2\, x_3 + 3x_4 \leq 0$
and $x_1, x_2, x_3, x_4, \geq 0$.
[Ans. Unbounded solution.]

5·8. SPECIAL CASES : ALTERNATIVE SOLUTIONS, UNBOUNDED SOLUTIONS AND NON-EXISTING SOLUTIONS

In this section, some important cases (except degeneracy) are discussed which are very often encountered during simplex procedure. The properties of these cases have already been visualised in the graphical solution of two variable LP problems.

5·8-1 Alternative Optimum Solutions

Example 22. *Use penalty (or Big-M) method to solve the problem :*

Max. $z = 6x_1 + 4x_2$, *subject to* $2x_1 + 3x_2 \leq 30$, $3x_1 + 2x_2 \leq 24$, $x_1 + x_2 \geq 3$, and $x_1, x_2 \geq 0$.

Is the solution unique ? If not, give two different solutions.

Solution. Introducing the slack variables $x_3 \geq 0$, $x_4 \geq 0$, surplus variable $x_5 \geq 0$, and artificial variable $a_1 \geq 0$, the problem becomes :

Max. $z = 6x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} 2x_1 + 3x_2 + x_3 \qquad\qquad &= 30 \\ 3x_1 + 2x_2 \qquad + x_4 \qquad &= 24 \\ x_1 + x_2 \qquad\qquad - x_5 + a_1 &= 3 \\ x_1, x_2, x_3, x_4, x_5, a_1 &\geq 0. \end{aligned}$$

Now the solution is obtained as follows :

Table 5·40

BASIC VARIABLES	C_B	X_B	X_1 (6)	X_2 (4)	X_3 (0)	X_4 (0)	X_5 (0)	A_1 (–M)	MIN RATIO (X_B/X_k)
x_3	0	30	2	3	1	0	0	0	30/2
x_4	0	24	3	2	0	1	0	0	24/3
$\leftarrow a_1$	$-M$	3	[1]	1	0	0	–1	1	3/1 ←
	$z = -3M$		$(-M-6)$ ↑	$(-M-4)$	0	0	M	0 ↓	$\leftarrow \Delta_j$
x_3	0	24	0	1	1	0	2	×	24/2
$\leftarrow x_4$	0	15	0	–1	0	1	[3]	×	15/3 ←
$\rightarrow x_1$	6	3	1	1	0	0	–1	×	—
	$z = 18$		0	2	0	0 ↓	–6 ↑	×	$\leftarrow \Delta_j$
$\leftarrow x_3$	0	14	0	[5/3]	1	– 2/3	0	×	$\frac{14}{5/3} = 42/5 \leftarrow$
$\rightarrow x_5$	0	5	0	– 1/3	0	1/3	1	×	—
x_1	6	8	1	2/3	0	1/3	0	×	$\frac{8}{2/3} = 12$
	$z = 48$		0	0* ↑	0 ↓	2	0	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, optimum solution is obtained as : $x_1 = 8$, $x_2 = 0$, max $z = 48$.

Alternative Solutions. Since Δ_2 corresponding to non-basic variable x_2 is obtained zero, this indicates that the *alternative solutions* also exist. Therefore, the solution as obtained above is not unique.

Thus we can bring $\mathbf{X}_2$ into the basis in place of $\mathbf{X}_3$. Therefore, introducing $\mathbf{X}_2$ into the basis in place of $\mathbf{X}_3$, the new optimum simplex table is obtained as follows :

Table 5·41

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_2	4	42/5	0	1	3/5	– 2/5	**0**	×	
x_5	0	39/5	0	0	1/5	1/5	**1**	×	
x_1	6	12/5	1	0	– 2/5	3/5	**0**	×	
	$z = 48$		0	0	0	2	0	×	$\leftarrow \Delta_j \geq 0$

From this table we get a different optimum solution : $x_1 = 12/5$, $x_2 = 42/5$, max. $z = 48$.

Thus, if two alternative optimum solutions can be obtained, then any number of optimum solutions can be obtained, as given below :

Variables	First Sol.	Second. Sol.	General Solution
x_1	8	12/5	$x_1 = 8\lambda + (12/5)(1-\lambda)$
x_2	0	42/5	$x_2 = 0\lambda + (42/5)(1-\lambda)$
x_3	14	0	$x_3 = 14\lambda + 0(1-\lambda)$
x_4	0	0	$x_4 = 0\lambda + 0(1-\lambda)$
x_5	5	39/5	$x_5 = 5\lambda + (39/5)(1-\lambda)$
a_1	0	0	$a_1 = 0\lambda + 0(1-\lambda)$

For any arbitrary value of λ, same optimal value of z will be obtained.

Note. If two optimum solutions of an LP problem are obtained. thus the mean of these two solutions will give us the third optimum solution. This process can be continued indefinitely to get as many alternative solutions as we want.

Example 23. *Maximize* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to the constraints :*

$x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$, *and* $x_1, x_2, x_3, x_4 \geq 0$.

Solution. Introducing artificial variables a_1 and a_2 in the *first* and *second* constraint equations, respectively, and the *original variable* x_4 can be treated to work as an artificial variable for the *third* constraint equation to obtain :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad + a_1 \quad &= 15 \\ 2x_1 + x_2 + 5x_3 \qquad + a_2 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \qquad &= 10 . \end{aligned}$$

Phase 1 : Table 5·42

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$
a_1	15	1	2	3	0	1	0
a_2	20	2	1	5	0	0	1
$\leftarrow x_4$	10	[1] ↑	2	1	1 ↓	0	0

By the same arguments as given in the previous examples of two-phase method insert $\mathbf{X_4}$ in place of $\mathbf{X_1}$. The transformed table (*Table 5·43*) is obtained by applying row transformations $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 2R_3$.

Table 5·43

BASIC VARIABES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$
a_1	5	0	0	2	−1	1	0
$\leftarrow a_2$	0	0	−3	[3]	−2	0	1
$\to x_1$	10	1	2	1 ↑	1	0	0 ↓

In spite of the fact that the artificial variable x_4 has served its purpose, the column $\mathbf{X_4}$ cannot be deleted from *Table 5·43,* because x_4 is the original variable also. Although the value of the artificial variable a_2 also becomes zero at this stage, the column $\mathbf{A_2}$ cannot be deleted unless it is inserted at one of the places $\mathbf{X_2}$ or $\mathbf{X_3}$ or $\mathbf{X_4}$ (wherever it is possible). Now, it is observed that $\mathbf{A_2}$ can be inserted in place of $\mathbf{X_3}$. Hence transformation *Table* 5·44 is obtained by applying the row transformations : $R_2 \to \frac{1}{3}R_2$, $R_1 \to R_1 - \frac{2}{3}R_2$, $R_3 \to R_3 - \frac{1}{3}R_2$.

Table 5·44

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$
$\leftarrow a_1$	5	0	2	0	1/3	1	−2/3
$\to x_3$	0	0	−1	1	−2/3	0	1/3
x_1	10	1	3	0	5/3	0	−4/3
			↑			↓	

Now removing $\mathbf{A_1}$ and inserting it in the suitable position of $\mathbf{X_2}$, the next transformed *Table 5·45* is obtained by row transformations : $R_1 \to \frac{1}{2}R_1$, $R_2 \to R_2 + \frac{1}{2}R_1$, $R_3 \to R_3 - \frac{3}{2}R_1$.

Table 5·45

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$
x_2	5/2	0	1	0	1/6	1/2
x_3	5/2	0	0	1	−1/2	1/2
x_1	5/2	1	0	0	7/6	−3/2

Delete column $\mathbf{A_1}$ $(a_1 = 0)$. The starting basic feasible solution is obtained : $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$. Further, proceed to test this solution for optimality in *Phase II.* For this, compute

$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (2, 3, 1)(1/6, -1/2, 7/6) - 0 = 0 .$$

Phase II. Table 5 46

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	Min. Ratio
x_2	2	5/2	0	1	0	1/6	
x_3	3	5/2	0	0	1	– 1/2	
x_1	1	5/2	1	0	0	7/6	
	$z = C_BX_B = 15$		0	0	0	0*	$\leftarrow \Delta_j$

Since all Δ_j 's are zero, the solution : $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$, is optimal to give us $z^* = 15$. Further, Δ_4 being zero indicates that *alternative* optimal solutions are also possible.

Note. Here Δ_j corresponding to nonbasic vector X_4 also becomes zero. This indicates that alternative optimum solutions are possible. However, the other optimal solutions can be obtained as : $x_1 = 0$, $x_2 = 15/7$, $x_3 = 25/7$, $x_4 = 0$, max. $z = 15$.

Now, given the *two* alternative basic solutions ;

(i) $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$ (ii) $x_1 = 0$, $x_2 = 15/7$, $x_3 = 25/7$, $x_4 = 0$

an infinite number of *non-basic* solutions can be obtained and by realizing them any weighted average of these two basic solutions is also an alternative optimum solution.

To verify this, *third* solution will be obtained as :

$$x_1 = \frac{5/2+0}{2},\ x_2 = \frac{5/2+15/7}{2},\ x_3 = \frac{5/2+25/7}{2},\ x_4 = \frac{0+0}{2}$$

i.e., $x_1 = 5/4$, $x_2 = 65/28$, $x_3 = 85/28$, $x_4 = 0$,

yielding the maximum value of $z = 15$.

Note. Also see example 14 (ch. 5, unit 2, page 88).

Example 24. *Following is the LP problem : Maximize* $z = x_1 + x_2 + x_4$, *subject to the constraints* :

$$x_1 + x_2 + x_3 + x_4 = 4,\ x_1 + 2x_2 + x_3 + x_5 = 4,\ x_1 + 2x_2 + x_3 = 4,\ x_1, x_2, x_3, x_4, x_5 \geq 0 .$$

(i) *Find out all the optimal basic feasible solutions by using penalty (or Big-M) method.*

(ii) *Write-down the general form of an optimal solution.*

Solution. Since the constraints of the given problem are already equations, only artificial variables are required to form the basis matrix. In order to bring the basis matrix as unit matrix, only artificial variable $a_1 \geq 0$ is needed in the third constraint. So the problem may be re-written in the form :

Max. $z = x_1 + x_2 + 0x_3 + x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 \quad\quad &= 4 \\ x_1 + 2x_2 + x_3 \quad + x_5 \quad &= 4 \\ x_1 + 2x_2 + x_3 \quad\quad + a_1 &= 4 \\ x_1, x_2, \ldots, x_5, a_1 &\geq 0 \end{aligned}$$

These constraints may be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} .$$

Applying the usual simplex method, the solution is obtained as follows :

Table 5·47

		$c_j \rightarrow$	1	1	0	1	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_4	1	4	1	1	1	1	0	0	4/1
x_5	0	4	1	2	1	0	1	0	4/2
$\leftarrow a_1$	$-M$	4	1	[2]	1	0	0	1	4/2 ←(Note)
	$z = -4M + 4$		$-M$	$-2M$ ↑	$-M+1$	0	0	0 ↓	$\leftarrow \Delta_j$

Note. Here it is observed that the minimum 4/2 occurs at two places (2nd and 3rd) in the last column. Although one of these two may be chosen by degeneracy rule (see 5.7-1, page 95), but minimum at 3rd place has been chosen to remove artificial basis vector A_1 from the basis matrix.

Table 5·48

		$c_j \rightarrow$	1	1	0	1	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_4	1	2	1/2	0	1/2	1	0	×	
x_5	0	0	0	0	0	0	1	×	
$\leftarrow x_2$	1	2	1/2	1	1/2	0	0	×	
	$z=4$		0*	0	1	0	0	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal basic feasible solution has been attained. Thus the optimum solution is given by

$$x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 2, x_5 = 0, \text{ max. } z = 4.$$

Since $\Delta_1 = 0$, alternative optimum solutions also exist.

5·8–2. Unbounded Solutions

The case of unbounded solutions occurs when the feasible region is unbounded such that the value of the objective function can be increased indefinitely. It is not necessary, however, that an unbounded feasible region should yield an unbounded value for the objective function. The following examples will illustrate these points.

Example 25. (Unbounded Optimal Solution)

Max. $z = 2x_1 + x_2$, *subject to* : $x_1 - x_2 \leq 10, 2x_1 - x_2 \leq 40$, *and* $x_1 \geq 0, x_2 \geq 0$.

Solution. The starting simplex table is as follows :

Table 5.49

BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1 (β_1)	S_2 (β_2)
s_1	0	10	1	−1	1	0
s_2	0	40	2	−1	0	1
	$z = C_B X_B = 0$		−2	−1	0	0

It can be seen from the starting simplex table that the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are candidates for the entering vector. Since Δ_1 has the minimum value, $\mathbf{X}_1$ should be selected as the entering vector. It is noticed, however, that if $\mathbf{X}_2$ is selected as the entering vector, the value of x_2 (and hence the value of z) can be increased indefinitely without affecting the feasibility of the solution (since it has all x_{i2} negative). It is thus concluded that the problem has no bounded solution. This can also be seen from the graphical solution of the problem in Fig. 5.1.

In general, *an unbounded solution can be detected if, at any iteration, any of the candidates for the entering vector* $\mathbf{X}_k$ *(for which* $\Delta_k < 0$, *i.e.* $z_k - c_k < 0$) *has all* $x_{ik} \leq 0$, $i = 1, 2, ..., m$, *i.e., all elements of the entering column are* ≤ 0.

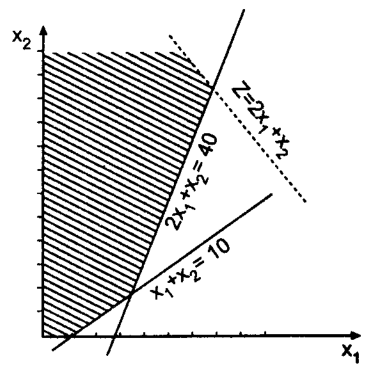


Fig. 5.1

Example 26. (Unbounded Solution)

Maximize $z = 107x_1 + x_2 + 2x_3$, *subject to* :

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7, 16x_1 + \tfrac{1}{2}x_2 - 6x_3 \leq 5, 3x_1 - x_2 - x_3 \leq 0, \textit{and } x_1, x_2, x_3 \geq 0.$$

Solution. By introducing slack variables, $x_5 \geq 0$, $x_6 \geq 0$, the set of constraints is converted into the system of equations :

$$\begin{cases} 14x_1 + x_2 - 6x_3 + 3x_4 = 7 \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 + x_5 = 5 \\ 3x_1 - x_2 - x_3 + x_6 = 0 \end{cases} \text{ or } \begin{cases} \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = 7/3 \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 + x_5 = 5 \\ 3x_1 - x_2 - x_3 + x_6 = 0 \end{cases}$$

or
$$\begin{bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}$$

Here original variable x_4 has been treated as slack variable as its coefficient in the objective function is zero, *i.e.*, *Maximize* $z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

Now start simplex method as follows :

Table 5.50

BASIC VARIABLES	C_B	X_B	X_1 (107)	X_2 (1)	X_3 (2)	X_4 (0)	X_5 (0)	X_6 (0)	MIN. RATIO
x_4	0	7/3	14/3	1/3	−2	1	0	0	7/14
x_5	0	5	16	1/2	−6	0	1	0	5/16
x_6	0	0	[3]	−1	−1	0	0	1	0/3 ←
	$z=0$		−107 ↑	−1	−2	0	0	0 ↓	← Δ_j
x_4	0	7/3	0	17/9	−4/9	1	0	−14/9	
x_5	0	5	0	35/6	−2/3	0	1	−16/3	
x_1	107	0	1	−1/3	−1/3	0	0	1/3	
	$z=0$		0	$-\frac{110}{3}$	$-\frac{113}{3}$	0	0	$\frac{107}{3}$	← Δ_j

Since corresponding to negative Δ_3 , all elements of X_3 column are negative, so X_3 cannot enter into the basis matrix. Consequently, this is an indication that there exists an *unbounded solution* to the given problem.

Example 27. ***(Unbounded feasible region but bounded optimal solution)***

Max. $z = 6x_1 - 2x_2$, *subject to* $2x_1 - x_2 \leq 2, x_1 \leq 4,$ *and* $x_1, x_2 \geq 0$.

Solution. We only give the successive tables here. Students are advised to fill up the details.

Table 5.51. Starting Simplex Table

BASIC VARIABLES	C_B	X_B	X_1 (6)	X_2 (−2)	X_3 (β_1) (0)	X_4 (β_2) (0)	MIN. RATIO (X_B/X_1)
x_3	0	2	← [2]	1	−1	0	2/2 ←
x_4	0	4	1	0	0	1	4/1
	$z = C_BX_B = 0$		−6 ↑	2	0 ↓	0	← Δ_j

First Improvement. We enter X_1 and remove β_1 .

Table 5.52

BASIC VARIABLES	C_B	X_B	X_1 (β_1) (6)	X_2 (−2)	X_3 (0)	X_4 (β_2) (0)	MIN. RATIO (X_B/X_2)
x_1	6	1	1	−1/2	1/2	0	—
x_4	0	3	0	← [1/2]	−1/2	−1	$3/\frac{1}{2}$ ←
	$z = C_BX_B = 6$		0	−1 ↑	3	0 ↓	← Δ_j

Second Improvement. Enter $\mathbf{X}_2$ and remove β_2.

Table 5·53

BASIC VARABLES	C_B	X_B	X_1 (β_1)	X_2 (β_2)	X_3	X_4	Min. Ratio
x_1	6	4	1	0	0	1	
x_2	–2	6	0	1	–1	2	
	$z = C_B X_B = 12$		0	0	2	2	$\leftarrow \Delta_j$

The optimal solution is : $x_1 = 4$, $x_2 = 6$, and $z = 12$.

It is now interesting to note from starting table that the elements of $\mathbf{X}_2$ are negative or zero (–1 and 0). This is an immediate indication that the feasible region is not bounded (see Fig. 5·2). From this, we conclude that a problem may have unbounded feasible region but still the optimal solution is bounded.

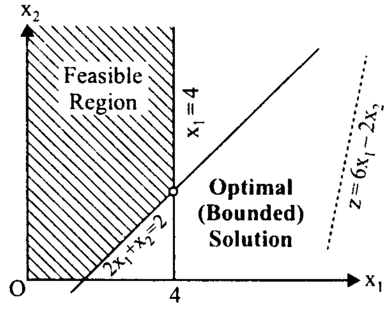


Fig. 5.2

5·8–3 . Non-existing feasible solutions

In this case, the feasible region is found to be empty which indicates that the problem has no feasible solution. The following example shows how such a situation can be detected by simplex method.

Example 28. ***(Problem with no feasible solution).***

Max. $z = 3x_1 + 2x_2$, *subject to* $2x_1 + x_2 \le 2$, $3x_1 + 4x_2 \ge 12$, *and* $x_1, x_2 \ge 0$.

[Garhwal 97; Meerut (O.R.) 90]

Solution. Introducing slack variable x_3, surplus variable x_4 together with the artificial variable a_1, the constraints become :

$$2x_1 + x_2 + x_3 = 2$$
$$3x_1 + 4x_2 - x_4 + a_1 = 12 .$$

Here we use M-technique for dealing with artificial variable a_1. For this, we write the objective function as

$$\text{Max. } z = 3x_1 + 2x_2 + 0x_3 + 0x_4 - Ma_1 .$$

The starting simplex table will be as follows.

Table 5·54

BASIC VARIABLES	C_B	X_B ($c_j \rightarrow$)	X_1 (3)	X_2 (2)	X_3 (β_1) (0)	X_4 (0)	A_1 (β_2) ($-M$)	MIN. RATIO (X_B/X_K)
$\leftarrow x_3$	0	2	1	[1]	1	0	0	2/1 ←
a_1	$-M$	12	3	4	0	–1	1	12/4
	$z = C_B X_B = -12M$		$(-3M-3)$	$(-4M-2)$ ↑	0 ↓	M	0	$\leftarrow \Delta_j$

$$\Delta_1 = C_B X_1 - c_1 = (0, -M)(2, 3) - 3 = (0 - 3M) - 3 = -3 - 3M$$
$$\Delta_2 = C_B X_2 - c_2 = (0, -M)(1, 4) - 2 = (0 - 4M) - 2 = -2 - 4M$$
$$\Delta_4 = C_B X_4 - c_4 = (0, -M)(0, -1) - 0 = M .$$

First improvement. Inserting $\mathbf{X}_2$ and removing β_1, *i.e.* $\mathbf{X}_3$

Table 5·55

BASIC VARIABLES	C_B	X_B ($c_j \rightarrow$)	X_1 (3)	X_2 (2)	X_3 (0)	X_4 (0)	A_1 ($-M$)	MIN. RATIO
x_2	2	2	2	1	1	0	0	
a_1	$-M$	4	–5	0	–4	–1	1	
	$z = C_B X_B = 4 - 4M$		$(1 + 5M)$	0	$(2 + 4M)$	M	0	$\leftarrow \Delta_j$

$$\Delta_1 = C_B Y_1 - c_1 = (2, -M)(2, -5) - 3 = (4 + 5M) - 3 = (1 + 5M)$$

$$\Delta_3 = \mathbf{C_B Y_3} - c_3 = (2\,, -M)\,(1\,, -4) - 0 = (2 + 4M) - 0 = (2 + 4M)$$
$$\Delta_4 = \mathbf{C_B Y_4} - c_4 = (2\,, -M)\,(0\,, -1) - 0 = (0 + M) = M\,.$$

Here all Δ_j are positive since $M > 0$. So according to the optimality condition, this solution is optimal.

Note. Here we should, however, note that the optimal (basic) solution :
$x_1 = 0\,,\ x_2 = 2\,,\ x_3 = 0\,,\ x_4 = 0\,,\ a_1 = 4\,,$
includes the artificial variable a_1 with positive value 4. This immediately indicates that the problem has no feasible solution, because the positive value of a_1 violates the second constraint of given problem. This situation can be observed by the graphical representation of this example in Fig. 5·3.

Such solution may be called *"pseudo-optimal"*, since (as clear from the Figure 5·3) it does not satisfy all the constraints, but it satisfies the optimality condition of the simplex method. **[JNTU (B.Tech) 98]**

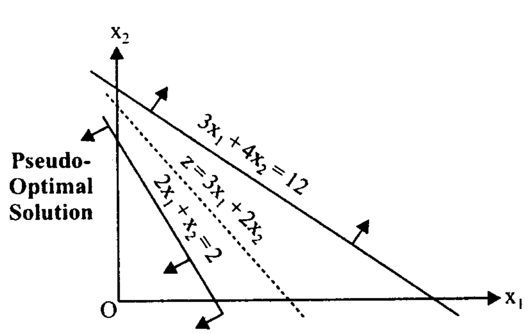


Fig. 5.3.

Q. 1. Show on graphy it (i) Unbounded solution space (ii) No feasible solution space. **[JNTU (B. Tech.) 2003]**

2. How are the following detected in simplex method ?
(i) Alternative solution (ii) Unbounded solution (iii) Infeasible solution. **[JNTU (B. Tech. III) 2003]**

5.9. BOUNDED VARIABLES PROBLEMS IN LP

In case the value of some or all variables is restricted with lower or upper bounds, the standard form of LP problem may be expressed as : Max (or Min) $z = \mathbf{cx}$, subject to $\mathbf{Ax} = \mathbf{b}, \boldsymbol{l} \le \mathbf{x} \le \mathbf{u}$ where $\boldsymbol{l} = (l_1, l_2, \ldots, l_n)$ and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ stand for *lower* and *upper* bounds for **x**, respectively. Other symbols have their usual meaning.

Then the constraint $\boldsymbol{l} \le \mathbf{x} \le \mathbf{u}$ can be converted into equality by introducing *slack* or *surplus* variables s' and s'' as follows :

$$\mathbf{x} \le \mathbf{u} \quad \Rightarrow \quad \mathbf{x} + s' = \mathbf{u},\ s' \ge 0$$

and $$\mathbf{x} \ge \boldsymbol{l} \quad \Rightarrow \quad \mathbf{x} - s'' = \boldsymbol{l},\ s'' \ge 0.$$

Therefore given *LP* has $m + n$ constraints as equations with $3n$ variables. However, we may also reduce this size in the form of $\mathbf{Ax} = \mathbf{b}$.

The lower bound constraint $\boldsymbol{l} \le \mathbf{x}$ can also be written as : $\mathbf{x} = \boldsymbol{l} + s'', s'' \ge 0$ and therefore **x** can be eliminated from all the constraints with the help of this substitution. Similarly, the upper bound $\mathbf{x} \le \mathbf{u}$ can also be written as : $\mathbf{x} = \mathbf{u} - s', s' \ge 0$. However, such substitution may not ensure non-negative value of **x**. Thus in order to overcome this difficulty, usual simplex method can also be modified to deal with bounded variables.

In bounded variable simplex method, the optimality condition remains the same. But the substitution of constraint $\mathbf{x} + \mathbf{s}' = \mathbf{u}$ in the simplex table require modification in the feasibility condition due to following reasons :

(i) A *basic variable* should become *non-basic* at its upper bound (in usual simplex method all *non-basic variables* are always *zero.*).

(ii) Whenever a non-basic variable becomes basic, its value should not be more than its upper bound and also should preserve the non-negativity and upper bound conditions of all basic variables that exist.

5.10. MODIFIED SIMPLEX ALGORITHM

The modified simplex algorithm for bounded variables can best be understood by following interative steps :

Step 1. If the objective function of the given problem is of minimization form, then change it to maximization form by using the relationship : Min $z = -$ Max $z', z' = -z$.

Step 2. Check whether all b_i, $i = 1, 2, \ldots, m$) are positive. If any of them is (are) negative. Then multiply the corresponding constraint by -1 in order to make it positive.

Step 3. Express the given *LP* problem in standard form by introducing slack/surplus variables.

Step 4. Obtain an Initial Basic Feasible Solution. If any of the basic variables is at a positive lower bound, then substitute it out at its lower bound.

Step 5. Compute $\Delta_j = C_B X_j - c_j, j = 1, 2, \ldots, n$ and examine the values of all Δ_j $(= z_j - c_j)$.

Step 6. (i) If all $\Delta_j \geq 0$, then the basic feasible solution is optimum.

(ii) If at least one of them is negative and the corresponding column has at least one entry positive (*i.e.*, $x_{ij} > 0$) for some row i, then it indicates that further improvement in z is possible.

Step 7. If case **(ii)** of ***step 6*** is possible, select a non-basic variable x_k to enter into new solution according to the most negative Δ_j $(= \Delta_k \uparrow)$.

Step 8. After identifying the non-basic variable (column vector) to enter the basis matrix **B**, the vector to be removed is determined. For this we first compute the quantities :

$$\theta_1 = \min_i \left\{ \frac{x_{Bi}}{x_{ir}}, x_{ir} > 0 \right\}, \theta_2 = \min_i \left\{ \frac{u_r - x_{Bi}}{-x_{ir}}, x_{ir} < 0 \right\}$$

and
$$\theta = \min\,[\theta_1, \theta_2, u_r]$$

where u_r is the upper bound for the variable x_r in the current basic feasible solution. Clearly, if all $x_{ir} > 0$, $\theta_2 = \infty$. Then following *three* possibilities may occur :

(i) If $\theta_1 = \theta$, then basis column vector $\mathbf{a}_k$ (basic varaible x_k) will be removed from the basis and thus it is replaced by column vector, say $\mathbf{a}_r$ (non-basic variable x_r) in the usual manner.

(ii) If $\theta_2 = \theta$, then basis column vector $\mathbf{a}_k$ (basic variable x_k) will be removed from the basis and is replaced by a column vector $\mathbf{a}_r$ (non-basic variable x_r). But the value of basic variable $x_r = x_{Br}$ will not be at upper bound at this stage. Now this must be substituted by using the relationship :

$$(x_{Bk})_r = (x_{Bk})'_r - x_{kr} u_r,\ 0 \leq (x_{Bk})_r' \leq u_r$$

where $(x_{Bk})'_r$ represents the value of x_r.

The value of non-basic variable x_r is given at its upper bound while the remaining non-basic variables are put at zero level by using the relationship : $x_r = u_r - x'_r,\ 0 \leq x'_r \leq u_r$.

Step 9. Go to ***step 7*** and repeat the procedure until all θ entries in Δ_j $(= z_j - c_j)$ row become either positive or zero.

5.11 ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following LP problem :*

Max $z = 3x_1 + 2x_2$, *subject to the constraints* :

$$x_1 - 3x_2 \leq 3,\ x_1 - 2x_2 \leq 4,\ 2x_1 + x_2 \leq 20,\ x_1 + 3x_2 \leq 30,\ -x_1 + x_2 \leq 6$$

and $0 \leq x_1 \leq 8,\ 0 \leq x_2 \leq 6.$

Solution. The given problem can be easily written in the following standard form :

$$\text{Max } z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 0s_5$$

subject to the constraints :

$$\begin{aligned} x_1 - 3x_2 + s_1 \quad\quad\quad\quad &= 3 \\ x_1 - 2x_2 \quad + s_2 \quad\quad\quad &= 4 \\ 2x_1 + x_2 \quad\quad + s_3 \quad\quad &= 20 \\ x_1 + 3x_2 \quad\quad\quad + s_4 \quad &= 30 \\ -x_1 + x_2 \quad\quad\quad\quad + s_5 &= 6 \end{aligned}$$

and
$$x_1, x_2, s_1, s_2, s_3, s_4, s_5 \geq 0.$$

Then the initial basic feasible solution is : $s_1 = 3$, $s_2 = 4$, $s_3 = 20$, $s_4 = 30$ and $s_5 = 6$. Since no upper bounds are given to these basic variables, we arbitrarily assume that all of them have upper bounds tending to ∞ (*i.e.*, $s_1 = s_2 = s_3 = s_4 = s_5 = \infty$). This solution may also be read from the following initial simplex *Table 5.56*.

Table 5.56 Initial Solution

$u_i \rightarrow$				8	6	∞	∞	∞	∞	∞	
$c_j \rightarrow$				3	2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
s_1	∞	0	3	[1]	−3	1	0	0	0	0	$(\infty - 3) \rightarrow \infty$
s_2	∞	0	4	1	−2	0	1	0	0	0	$(\infty - 4) \rightarrow \infty$
s_3	∞	0	20	2	1	0	0	1	0	0	$(\infty - 20) \rightarrow \infty$
s_4	∞	0	30	1	3	0	0	0	1	0	$(\infty - 30) \rightarrow \infty$
s_5	∞	0	6	−1	1	0	0	0	0	1	$(\infty - 6 \rightarrow \infty)$
	$z = 0$			[−3] ↑	−2	0	0	0	0	0	$\leftarrow \Delta_j$

Since $\Delta_1 = z_1 - c_1 = -3$ is largest negative, x_1 will *enter into the basis.*

As none of the basic variables s_1 to s_5 are at their upper bound, thus to decide about the variable to *leave the basic solution* we calculate.

$$\theta_1 = \underset{i}{\text{Min}}\left(\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right) = \text{Min}\left(\frac{3}{1}, \frac{4}{1}, \frac{20}{2}, \frac{30}{1}\right) = 3 \qquad \text{(corresponding to } x_1)$$

$$\theta_2 = \underset{i}{\text{Min}}\left(\frac{u_i - x_{Bi}}{-x_{i1}}, x_{i1} < 0\right) = \text{Min}\left(\frac{\infty - 6}{-(-1)}\right) = \infty \qquad \text{(corresponding to } s_5)$$

and $u_1 = 8$.

$$\therefore \qquad \theta = \text{Min}\ (\theta_1, \theta_2, u_1) = \text{Min}\ (3, \infty, 8) = 3 \qquad \text{(corresponding to } \theta_1)$$

Hence the non-basic variable s_1 is eligible to leave the basic solution and therefore $\boxed{x_{11} = 1}$ will become the *key element.*

Thus introducing x_1 into the basis and *removing* s_1 from the basis using row operations in an usual manner we get the improved solution *Table 5.57*

Table 5.57

$u_i \rightarrow$				8	6	∞	∞	∞	∞	∞	
$c_j \rightarrow$				3	2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
x_1	8	3	3	1	−3	1	0	0	0	0	$(8 - 3) \rightarrow 5$
s_2	∞	0	1	0	[1]	−1	1	0	0	0	$(\infty - 1) \rightarrow \infty$
s_3	∞	0	14	0	7	−2	0	1	0	0	$(\infty - 14) \rightarrow \infty$
s_4	∞	0	27	0	6	−1	0	0	1	0	$(\infty - 27) \rightarrow \infty$
s_5	∞	0	9	0	−2	1	0	0	0	1	$(\infty - 9) \rightarrow \infty$
	$z = 9$			0	−11↑	3	0	0	0	0	$\leftarrow \Delta_j$

Since $\Delta_2 = -11$ is the largest negative quantity, variable x_2 must enter the basis. So, to find the variable to leave the basis, we compute

$$\theta_1 = \underset{i}{\text{Min}}\left[\frac{x_{Bi}}{x_{i2}}, x_{i2} > 0\right] = \text{Min}\left[\frac{1}{1}, \frac{14}{7}, \frac{27}{6}\right] = 1 \qquad \text{(corresponding to } x_2)$$

$$\theta_2 = \underset{i}{\text{Min}}\left[\frac{u_i - x_{Bi}}{-x_{i2}}, x_{i2} < 0\right] = \text{Min}\left[\frac{8-3}{-(-3)}, \frac{\infty}{-(-2)}\right] = \frac{5}{3} \qquad \text{(corresponding to } x_1)$$

and $\quad u_2 = 6$.

Therefore, $\theta = \text{Min}\ [\theta_1, \theta_2, u_2] = \text{Min}\ [1, 5/3, 6] = 1$ (corresponding to s_2)

Hence s_2 must leave the basis and $x_{22} = 1$ will be the key element. So introduce x_2 into the basis and remove s_2 from the basis in the usual manner. The improved solution is given in *Table 5.58.*

Table 5.58

			$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞	
			$c_j \rightarrow$	3	2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
x_1	8	3	6	1	0	[−2]	3	0	0	0	8 − 6 = 2
x_2	6	2	1	0	1	−1	1	0	0	0	6 − 1 = 5
s_3	∞	0	7	0	0	5	−7	1	0	0	∞ − 7 → ∞
s_4	∞	0	21	0	0	5	−6	0	1	0	∞ − 21 → ∞
s_5	∞	0	11	0	0	−1	2	0	0	1	∞ − 11 → ∞
	z = 20			0	0	−8 ↑	11	0	0	0	← Δ_j

In this table, $\Delta_3 = z_3 - c_3 = -8$ is the largest negative quantity. Therefore s_1 will *enter into the basis.* Now to find the variable to be removed from the basis we calculate :

$$\theta_1 = \underset{i}{\text{Min}}\left(\frac{x_{Bi}}{x_{i3}}, x_{i3} > 0\right) = \text{Min}\left(\frac{7}{5}, \frac{21}{5}\right) = \frac{7}{5} \qquad \text{(corresponding to } s_3\text{)}$$

$$\theta_2 = \underset{i}{\text{Min}}\left(\frac{u_i - x_{Bi}}{-x_{i1}}, x_{i1} < 0\right) = \text{Min}\left(\frac{8-6}{-(-2)}, \frac{6-1}{-(-1)}, \frac{\infty}{-(-1)}\right) = 1 \qquad \text{(corresponding to } x_1\text{)}$$

and $u_3 = \infty$.

$\therefore$ $\theta = \text{Min}\ (\theta_1, \theta_2, u_3) = \text{Min}\ (7/5, 1, \infty) = 1$ (corresponding to x_1)

Therefore, the variable x_1 will leave the basis and $x_{13} = -2$ will become the key element.

Thus introducing s_1 into the basis and removing x_1 from the basis by row transformation, we set the following improved solution *Table 5.59.*

Table 5.59

			$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞	
			$c_j \rightarrow$	3	2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
s_1	∞	0	−3	−1/2	0	1	−3/2	0	0	0	∞ − (−3) → ∞
x_2	6	2	−2	−1/2	1	0	−1/2	0	0	0	6 − (−2) = 8
s_3	∞	0	22	5/2	0	0	1/2	1	0	0	∞ − 22 → ∞
s_4	∞	0	36	5/2	0	0	3/2	0	1	0	∞ − 36 → ∞
s_5	∞	0	8	−1/2	0	0	1/2	0	0	1	∞ − 8 → ∞
	z = −4			−4 ↑	0	0	−1	0	0	0	← Δ_j

Since $\Delta_1 = z_1 - c_1 = -4$ is the largest negative quantity, x_1 will enter into the basis. But upper bound for the variable x_1 is 8, therefore we can update the value of basic variables by using the relations and data of *Table 5.59.*

$$s_1 = x_{B1} = x'_{Bi} - x_{11}\ u_1 = -3 - (-1/2)\ 8 = 1$$
$$x_2 = x_{B2} = x'_{B2} - x_{21}\ u_1 = -2 - (-1/2)\ 8 = 2$$
$$s_3 = x_{B3} = x'_{B3} - x_{31}\ u_1 = 22 - (5/2)\ 8 = 2$$
$$s_4 = x_{B4} = x'_{B4} - x_{41}\ u_1 = 36 - (5/2)\ 8 = 16$$
$$s_5 = x_{B5} = x'_{B5} - x_{51}\ u_1 = 8 - (-1/2)\ 8 = 12.$$

Also, the non-basic variable x_1 having its upper bound can be made non-basic (*i.e.* zero) by using the substitution

$$x_1 = u_1 - x'_1 = 8 - x'_1,\ 0 \le x'_1 \le 8.$$

Now we can update the data of *Table* 5.59 by substituting new values of basic variables as well as non-basic variables as given below in *Table* 5.60.

Table 5.60

			$u_j \to$	8	6	∞	∞	∞	∞	∞	
			$c_j \to$	−3	2	0	0	0	0	0	
Basic Var.	u_B	c_B	x_B	X'_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
s_1	∞	0	1	1/2	0	1	−3/2	0	0	0	∞ − 1 → ∞
x_2	6	2	2	1/2	1	0	−1/2	0	0	0	6 − 2 = 4
s_3	∞	0	2	−5/2	0	0	[1/2]	1	0	0	∞ − 2 → ∞
s_4	∞	0	16	−5/2	0	0	3/2	0	1	0	∞ − 16 → ∞
s_5	∞	0	12	1/2	0	0	1/2	0	0	1	∞ − 12 → ∞
	$z = 24 + 4 = 28$			4	0	0	−1 ↑	0	0	0	← Δ_j

Since Δ_4 only is negative (*i.e.*, − 1), s_2 will *enter into the basis.* To find the variable to be *removed from the basis,* we calculate

$$\theta_1 = \underset{i}{\text{Min}}\left(\frac{x_{Bi}}{x_{i4}}, x_{i4} > 0\right) = \text{Min}\left(\frac{2}{1/2}, \frac{16}{3/2}, \frac{12}{1/2}\right) = \text{Min}\ (4, 32/3, 24) = 4 \text{ (corresponding to } s_3)$$

$$\theta_2 = \underset{i}{\text{Min}}\left(\frac{u_i - x_{Bi}}{-x_{i4}}, x_{i4} < 0\right) = \text{Min}\left(\frac{\infty}{-(-3/2)}, \frac{6-2}{-(-1/2)}\right) = 8 \qquad \text{(corresponding to } x_2)$$

and $\quad u_4 = \infty.$

$\therefore \quad \theta = \text{Min}\ (\theta_1, \theta_2, u_4) = \text{Min}\ (4, 8, \infty) = 4 \qquad$ (corresponding to s_3).

Therefore, the basic variable s_3 will be removed from the basis and thus $x_{34} = 1/2$ will become the key element. Thus *introducing* s_2 into the basis and *removing* s_3 from the basis, we get the following improved solution *Table* 5.61.

Table 5.61

			$u_i \to$	8	6	∞	∞	∞	∞	∞	
			$c_j \to$	−3	2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X'_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
s_1	∞	0	7	−7	0	1	0	3	0	0	∞ − 1 → ∞
x_2	6	2	4	[−2]	1	0	0	1	0	0	6 − 4 = 2
s_2	∞	0	4	−5	0	0	1	2	0	0	∞ − 4 → ∞
s_4	∞	0	10	5	0	0	0	−3	1	0	∞ − 10 → ∞
s_5	∞	0	10	3	0	0	0	−1	0	1	∞ − 10 → ∞
	$z = 28 + 4 = 32$			−1	0	0	0	2	0	0	← Δ_j

Since Δ_1 is the only negative value, variable x_1 will enter into the basis. To find the variable to be removed from the basis, we calculate

$$\theta_1 = \underset{i}{\text{Min}}\left(\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right) = \text{Min}\left(\frac{10}{5}, \frac{10}{3}\right) = \frac{10}{5} \qquad \text{(corresponding to } s_4)$$

$$\theta_2 = \underset{i}{\text{Min}}\left(\frac{u_i - x_{Bi}}{-x_{i1}}, x_{i1} < 0\right) = \text{Min}\left(\frac{\infty}{-(-7)}, \frac{6-4}{-(-2)}, \frac{\infty}{-(-5)}\right) = 1 \qquad \text{(corresponding to } x_2)$$

and $\quad u_1 = 8.$

$\therefore \quad \theta = \text{Min}\ (\theta_1, \theta_2, u_1) = \left(\frac{12}{5}, 1, 8\right) = 1 \qquad$ (corresponding to x_2)

Thus, the variable x_2 will leave the basis and $x_{21} = -2$ will become the key element. Hence introducing x'_1 into the basis and removing x_2 from the basis, we get the following improved solution *Table* 5.62.

Table 5.62

	$u_j \rightarrow$			8	6	∞	∞	∞	∞	∞	
	$c_j \rightarrow$			3	2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X'_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
s_1	∞	0	−7	0	−7/2	1	0	−1/2	0	0	$\infty + 7 \to \infty$
x'_1	8	−3	−2	1	−1/2	0	0	−1/2	0	0	$-3 + 2 = -1$
s_2	∞	0	−6	0	−5/2	0	1	−1/2	0	0	$\infty + 6 \to \infty$
s_4	∞	0	22	0	5/2	0	0	−1/2	1	0	$\infty - 22 \to \infty$
s_5	∞	0	16	0	3/2	0	0	1/2	0	1	$\infty - 16 \to \infty$
	$z = 32 - 2 = 30$			0	−1/2	0	0	3/2	0	0	$\leftarrow \Delta_j$

Since the upper bound for the variable x_2 is 6, we can update the basic variables by the following relations and the data of *Table* 10.7.

$$s_1 = x_{B1} = x'_{B1} - x_{12}u_2 = -7 - (-7/2)\,6 = 14$$
$$x'_1 = x_{B2} = x'_{B2} - x_{22}u_2 = -2 - (1/2)\,6 = 1$$
$$s_2 = x_{B3} = x'_{B3} - x_{32}u_2 = -6 - (-5/2)\,6 = 9$$
$$s_4 = x_{B4} = x'_{B4} - x_{42}u_2 = -22 - (5/2)\,6 = 7$$
$$s_5 = x_{B5} = x'_{B5} - x_{52}u_2 = 16 - (3/2)\,6 = 7.$$

Therefore, the non-basic variable x_2 at its upper bound can be made zero by using the substitution

$$x_2 = u_2 - x'_2 = 6 - x'_2,\ 0 \le x'_2 \le 6.$$

Now the data of *Table* 5.62 can be updated by substituting new values of basic variables and non-basic variables as given below in *Table* 5.63.

Table 5.63

		$u_j \rightarrow$		8	6	∞	∞	∞	∞	∞	
		$c_j \rightarrow$		−3	−2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X'_1	X'_2	S_1	S_2	S_3	S_4	S_5	
s_1	∞	0	14	0	7/2	1	0	−1/2	0	0	
x'_1	8	−3	1	1	1/2	0	0	−1/2	0	0	
s_2	∞	0	9	0	5/2	0	1	−1/2	0	0	
s_4	∞	0	7	0	−5/2	0	0	−1/2	1	0	
s_5	∞	0	7	0	−3/2	0	0	1/2	0	1	
	$z = 12 + 21 = 33$			0	1/2	0	0	3/2	0	0	$\leftarrow \Delta_j$

Since all $\Delta_j \ge 0$, an optimum solution is obtained with the values of variables as :

$$x'_1 = 1 \text{ or } x_1 = u_1 - x'_1 = 8 - 1 = 7; x_2 = u_2 - x'_2 = 6 - 0 = 6 \text{ and max } z = 33.$$

Example 2. *Solve the following LP problem :*

Maximize $z = 3x_1 + 5x_2 + 2x_3$, *subject to*

$$x_1 + 2x_2 + 2x_3 \le 14$$
$$2x_1 + 4x_2 + 3x_3 \le 23$$
$$0 \le x_1 \le 4$$
$$2 \le x_2 \le 5$$
$$0 \le x_3 \le 3.$$

Solution. The variable x_2 has a positive lower bound, therefore we can take the substitution $x'_2 = x_2 - 2$ or $x_2 = x'_2 + 2$. Thus the fourth constraint of the given problem can be written as $0 \le x'_2 \le 3$ and thus new LP problem becomes :

$$\text{Maximize} \quad z = 3x_1 + 5(x'_2 + 2) + 2x_3 = 3x_1 + 5x'_2 + 2x_3 + 10, \text{ subject to}$$
$$x_1 + 2(x'_2 + 2) + 2x_3 \le 14 \text{ or } x_1 + 2x'_2 + 2x_3 \le 10$$
$$2x_1 + 4(x'_2 + 2) + 3x_3 \le 23 \text{ or } 2x_1 + 4x'_2 + 3x_3 \le 15$$
$$0 \le x_1 \le 4$$
$$0 \le x'_2 \le 3$$
$$0 \le x_3 \le 3$$

with the help of non-negative slack variables s_1 and s_2, inequality constraints are converted to equations and thus standard form of LP problem becomes :

$$\text{Max. } z = 3x_1 + 5x'_2 + 2x_3 + 0s_1 + 0s_2 + 10, \text{ subject to the constraints}$$
$$x_1 + 2x'_2 + 2x_3 + s_1 \quad = 10$$
$$2x_1 + 4x'_2 + 3x_3 \quad + s_2 = 15$$

and $\qquad x_1, x'_2, x_3, s_1, s_2 \ge 0.$

The initial basic feasible solution becomes

$$s_1 = x_{B1} = 10, \; s_2 = x_{B2} = 15.$$

But, there are no upper bounds for the basic variables s_1 and s_2. Therefore, we may assume that both of these varaibles have an upper bound at ∞. Thus the initial basic feasible solution can be read from the initial simplex *Table* 5.64.

Table 5.64

			$u_j \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	3	5	2	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X'_2	X_3	S_1	S_2	$u_i - x_{Bi}$
s_1	∞	0	10	1	2	2	1	0	$\infty - 10 \rightarrow \infty$
s_2	∞	0	15	2	[4]	3	0	1	$\infty - 15 \rightarrow \infty$
	$z = 10$			-3	-5 ↑	-2	0	0	$\leftarrow \Delta_j$

Since $\Delta_2 = -5$ is the most negative, quantity x'_2 will enter into the basis. Also none of the basic variables (s_1 and s_2) are at their upper bounds. Therefore, to decide about the leaving variable, we calculate

$$\theta_1 = \underset{i}{\text{Min}}\left(\frac{x_{Bi}}{x_{i2}}, x_{i2} > 0\right) = \text{Min}\left(\frac{10}{2}, \frac{15}{4}\right) = \frac{15}{4} \qquad \text{(corresponding to } s_2\text{)}$$

$$\theta_2 = \infty. \qquad (\because \text{ all entries in 2nd column are + ive, } x_{i2} > 0 \text{ for } i)$$

and $\qquad u_2 = 3.$

$$\therefore \qquad \theta = \text{Min}(\theta_1, \theta_2, u_2) = \text{Min}\left(\frac{15}{4}, \infty, 3\right) = 3 \qquad \text{(corresponding to } u_2\text{)}$$

Thus, the non-basic variable x'_2 can be substituted at its upper bound and will remain non-basic. Now the non-basic variable x'_2 at its upper bound will have the value zero by substitution :

$$x'_2 = u_2 - x''_2 = 3 - x''_2, \; 0 \le x''_2 \le 3.$$

The value of basic variables can be updated by using

$$s_1 = x_{B1} = x'_{B1} - x_{12}u_2 = 10 - 2 \times 3 = 4$$
$$s_2 = x_{B2} = x'_{B2} - x_{22}\,u_2 = 15 - 4 \times 3 = 3.$$

Now the *Table* 5.64 can be updated by substituting these new values of basic variables and non-basic variable x'_2 as given below in *Table* 5.65.

Table 5.65

			$u_i \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	3	−5	2	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X''_2	X_3	s_1	s_2	$u_i - x_{Bi}$
s_1	∞	0	4	1	−2	2	1	0	∞ − 4 → ∞
s_2	∞	0	3	[2]	−4	3	0	1	∞ − 3 → ∞
	$z = 10 + 15 = 25$			−3	5	−2	0	0	← Δ_j

In this table, $\Delta_1 = -3$ is the most negative value, therefore x_1 will be the ***entering variable*** into the basis. Then, to decide about the variable *leaving* from the basis, we calculate

$$\theta_1 = \text{Min}\left(\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right) = \text{Min}\left(\frac{4}{1}, \frac{3}{2}\right) = 3/2 \qquad \text{(corresponding to } s_2)$$

$$\theta_2 = \infty \qquad (\because \text{ all entries in 1st column are + ive, } i.e.\ x_{i1} > 0\ \forall\ i)$$

and $\quad u_1 = 4.$

$\therefore \quad \theta = \text{Min}(\theta_1, \theta_2, u_1) = \text{Min}(3/2, \infty, 4) = 3/2 \qquad$ (corresponding to θ_1)

Hence basic variable s_2 will leave the basis and then $x_{21} = 2$ will become the ***key element.*** Therefore, introducing x_1 into the basis and removing s_2 from the basis in usual manner, we get the following improved solution *Table* 5.66.

Table 5.66

			$u_j \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	3	−5	2	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X''_2	X_3	S_1	S_2	$u_i - x_{Bi}$
s_1	∞	0	5/2	0	0	1/2	1	−3/2	∞ − 5/2 → ∞
x_1	3	3	3/2	1	[−2]	3/2	0	1/2	3 − 3/2 = 3/2
	$z = 25 + 9/2 = 59/2$			0	−1 ↑	5/2	0	3/2	← Δ_j

In this table, $\Delta_2 = -1$ is the only negative value, therefore x''_2 will be the variable *entering* into the basis. Then to decide about the variable *leaving* from the basis, we calculate.

$$\theta_1 = \infty \qquad (\because \text{ All entries in 2nd column are either 0 or negative, } i.e.\ x_{i2} \le 0\ \forall\ i)$$

$$\theta_2 = \text{Min}\left(\frac{u_i - x_{Bi}}{-x_{i2}}, x_{i2} < 0\right) = \text{Min}\left(\infty, \frac{3/2}{-(-2)}\right) = \frac{3}{4} \qquad \text{(corresponding to } x_1)$$

and $\quad u_2 = 3.$

$\therefore \quad \theta = \text{Min}(\theta_1, \theta_2, u_2) = \text{Min}(\infty, 3/4, 3) = 3/4 \qquad$ (corresponding to θ_2)

Thus, variable x_1 will leave the basis. Substituting $x_1 = 4 - x'_1$ in *Table* 5.65 for putting x_1 at its upper bound, we get the following improved solution *Table* 5.67.

Table 5.67

			$u_i \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	3	−5	2	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X''_2	X_3	S_1	S_2	
s_1	∞	0	5/2	0	0	1/2	1	−1/2	
x''_2	3	−5	−3/4	−1/2	1	−3/4	0	−1/4	
	$z = 25 + 15/4$			−1/2 ↑	0	7/4	0	5/4	← Δ_j

The value of non-basic variable x_1 at its upper bound 4 can be put by substituting

$$x_1 = 4 - x'_1,\ 0 \le x'_1 \le 4.$$

The values of other basic variables can be updated by using

$$s_1 = x_{B1} = x'_{B1} - x_{11}u_1 = 5/2 - 0 \times 4 = 5/2$$

$$x''_2 = x_{B2} = x'_{B2} - x_{21}\,u_1 = -3/4 - (-1/2) \times 4 = 5/4$$

Now the *Table* 5.67 can be updated by taking new values of basic variables and non-basic x'_1 as shown below in *Table* 5.68.

Table 5.68

			$u_j \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	−3	−5	2	0	0	
Basic Var.	U_B	C_B	X_B	X'_1	X''_2	X_3	S_1	S_2	
s_1	∞	0	5/2	0	0	1/2	1	−1/2	
x''_2	3	−5	5/4	1/2	1	−3/4	0	−1/4	
	$z = 123/4$			1/2	0	7/4	0	5/4	$\leftarrow \Delta_j$

Since all $\Delta_j \ge 0$, the optimal solution becomes :

$$x'_1 = 0 \quad \text{or} \quad 4 - x_1 = 0 \quad \text{or} \quad x_1 = 4$$

$$x''_2 = 5/4 \quad \text{or} \quad 3 - x'_2 = 5/4 \quad \text{or} \quad 3 - (x_2 - 2) = 5/4 \quad \text{or} \quad x_2 = 15/4$$

and $\max z = 123/4$.

EXAMINATION PROBLEMS

Solve the following LP problems :

1. Max. $z = 2x_1 + x_2$, subject to the constraints :
   $x_1 + 2x_2 \le 10,\ x_1 + x_2 \le 6,\ x_1 - x_2 \le 2,\ x_1 - 2x_2 \le 1,\ 0 \le x_1 \le 3,\ 0 \le x_2 \le 2.$
   [**Ans.** $x_1 = 3,\ x_2 = 2$ and max $z = 8$]
2. Max $z = x_2 + 3x_3$, subject to
   $x_1 + x_2 + x_3 \le 10,\ x_1 - 2x_3 \le 0,\ 0 \le x_1 \le 8,\ 0 \le x_2 \le 4,\ x_3 \le 0.$
   [**Ans.** $x_1 = 20/3,\ x_2 = 0,\ x_3 = 10/3$ and max. $z = 10$]
3. Max $z = x_1 + x_2 + 3x_3$, subject to $x_1 + x_2 + x_3 \le 12,\ -x_1 + x_2 \le 5,\ x_2 + 2x_3 \le 8,$
   $0 \le x_1 \le 3,\ 0 \le x_2 \le 6,\ 0 \le x_3 \le 4.$
4. Max. $z = 4x_1 + 4x_2 + 3x_3$, subject to
   $-x_1 + 2x_2 + 3x_3 \le 15,\ -x_2 + x_3 \le 4,\ 2x_1 + x_2 - x_3 \le 6,\ x_1 - x_2 + 2x_3 \le 10,$
   $0 \le x_1 \le 8,\ 0 \le x_2 \le 4,\ 0 \le x_3 \le 4.$
   [**Ans.** $x_1 = 17/5,\ x_2 = 16/5,\ x_3 = 4$ and Max. $z = 192/5$]
5. Max. $z = 2x_1 + 3x_2 - 2x_3$, subject to
   $x_1 + x_2 + x_3 \le 8,\ 2x_1 + x_2 - x_3 \ge 3,\ 0 \le x\,x_1 \le 4,\ -2 \le x_2 \le 6,\ x_3 \ge 2$
6. Min. $z = x_1 + 2x_2 + 3x_3 - x_4$, subject to
   $x_1 - x_2 + x_3 - 2x_4 \le 6,\ -x_1 + x_2 - x_3 + x_4 \le 8,\ 2x_1 + x_2 - x_3 \le 2,$
   $0 \le x_1 \le 3,\ 1 \le x_2 \le 4,\ 0 \le x_3 \le 10,\ 2 \le x_4 \le 5.$
7. Max. $z = 4x_1 + 10x_2 + 9x_3 + 11x_4$, subject to
   $2x_1 + 2x_2 + 2x_3 + 2x_4 \le 5$
   $48x_1 + 80x_2 + 160x_3 + 240x_4 \le 257$
   and $0 \le x_j \le 1,\ j = 1, 2, 3, 4.$
   [**Ans.** $x_1 = 9/16,\ x_2 = 1,\ x_3 = 15/16,\ x_4 = 0$ and Max. $z = 331/16$]
8. Max $z = 3x_1 + x_2 + x_3 + 7x_4$ subject to
   $2x_1 + 3x_2 - x_3 + 4x_4 \le 40,\ -2x_1 + 2x_2 + 5x_3 - x_4 \le 35,$
   $x_1 + x_2 - 2x_3 + 3x_4 \le 100$ and $x_1 \ge 2,\ x_2 \ge 1,\ x_3 \ge 3,\ x_4 \ge 4.$
   [**Ans.** $x_1 = 71/4,\ x_2 = 1,\ x_3 = 29/2,\ x_4 = 4$, Max. $z = 287/4$].
9. Min. $z = -2x_1 - 4x_2 - x_3$, subject to
   $2x_1 + x_2 + x_3 \le 10,\ x_1 + x_2 - x_3 \le 4$ and
   $0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ 1 \le x_3 < 4.$
   [**Ans.** $x_1 = 2/3,\ x_2 = 6,\ x_3 = 8/3$ and Min. $z = -28$.]

5·12. SOLUTION OF SIMULTANEOUS EQUATIONS BY SIMPLEX METHOD

For the solution of n simultaneous linear equations in n variables a *dummy* objective function is introduced as

$$\text{Max. } z = 0\,\mathbf{X} - 1\,\mathbf{X}_a$$

where $\mathbf{X}_a$ are artificial variables, and $x_r = x_r' - x_r''$, such that $x_r' \geq 0$, $x_r'' \geq 0$.

The reformulated linear programming problem is then solved by simplex method. The optimal solution of this problem gives the values of the variables (**X**).

The following example will illustrate the procedure.

Example 29. *Use simplex method to solve the following system of linear equations :*

$$x_1 - x_3 + 4x_4 = 3\,,\; 2x_1 - x_2 = 3\,,\; 3x_1 - 2x_2 - x_4 = 1\,,\text{ and } x_1, x_2, x_3, x_4 \geq 0\,.$$

[Meerut M. Com. Jan. 98 (BP)]

Solution. Since the objective function for the given constraint equation is not prescribed, so a dummy objective function is introduced as :

Max. $z = 0x_1 + 0x_2 + 0x_3 + 0x_4 - 1a_1 - 1a_2 - 1a_3$, where $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$ are artificial variables.
Introducing artificial variables, the given equations can be written as :

$$\begin{aligned} x_1 \quad - x_3 + 4x_4 + a_1 \qquad &= 3 \\ 2x_1 - x_2 \qquad\qquad + a_2 \quad &= 3 \\ 3x_1 - 2x_2 \quad - x_4 \qquad + a_3 &= 1\,. \end{aligned}$$

Now apply simplex method to solve the reformulated problem as shown in *Table 5·69.*

Table 5·69

BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	A_3	MIN RATIO (X_B/X_k)
$c_j \rightarrow$			0	0	0	0	−1	−1	−1	
a_1	−1	3	1	0	−1	4	1	0	0	3/1
a_2	−1	3	2	−1	0	0	0	1	0	3/2
$\leftarrow a_3$	−1	1	[3]	−2	0	−1	0	0	1	1/3 ←
	$z = -10$		−6 ↑	3	1	−3	0	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	−1	8/3	0	2/3	−1	[13/3]	1	0	×	8/13 ←
a_2	−1	7/3	0	1/3	0	2/3	0	1	×	7/2
$\rightarrow x_1$	0	1/3	1	−2/3	0	−1/3	0	0	×	—
	$z = -5$		0	−1	1	−5 ↑	0 ↓	0	×	$\leftarrow \Delta_j$
$\leftarrow x_4$	0	8/13	0	[2/13]	−3/13	1	×	0	×	8/2 ←
a_2	−1	25/13	0	3/13	2/13	0	×	1	×	25/3
x_1	0	7/13	1	−8/13	1/13	0	×	0	×	—
	$z = 25/13$		0	−3/13 ↑	−2/13	0 ↓	×	0	×	$\leftarrow \Delta_j$
$\rightarrow x_2$	0	4	0	1	−3/2	13/2	×	0	×	—
$\leftarrow a_2$	−1	1	0	0	← [1/2]	3/2	×	1	×	$1/\frac{1}{2}$ ←
x_1	0	3	1	0	−1	4	×	0	×	—
	$z = -1$		0	0	−1/2 ↑	3/2	×	0 ↓	×	$\leftarrow \Delta_j$
x_2	0	7	0	1	0	2	×	×	×	
$\rightarrow x_3$	0	2	0	0	1	−3	×	×	×	
x_1	0	5	1	0	0	1	×	×	×	
	$z = 0$		0	0	0	0	×	×	×	$\leftarrow \Delta_j = 0$

Since all $\Delta_j = 0$, an optimum solution has been attained. Thus the solution of simultaneous equations is given by, $x_1 = 5$, $x_2 = 7$, $x_3 = 2$, and $x_4 = 0$.

5.13. INVERSE OF A MATRIX BY SIMPLEX METHOD

Let A be any $n \times n$ real matrix. Let $\mathbf{X} \in R^n$ and $\mathbf{b}$ be any dummy $n \times 1$ real matrix. Then, consider the system of equations : $\mathbf{AX} = \mathbf{b}$; $\mathbf{X} \geq \mathbf{0}$.

By introducing a *dummy* objective function $z = 0\,\mathbf{X} - 1\mathbf{X}_a$, where $\mathbf{X}_a \geq 0$ being artificial variable vector, then we find a solution to the LPP of maximizing z subject to the constraints : $\mathbf{AX} + 1\,\mathbf{X}_a = \mathbf{b}$; $\mathbf{X}, \mathbf{X}_a \geq \mathbf{0}$.

If we get the optimal solution to the given LPP in which the basis contains all the variables of vector $\mathbf{X}$, then inverse of A is directly read off from the optimum (final) simplex table. Then inverse of A consists of those column vectors in the last iteration of the simplex method which were present in the initial basis $\mathbf{B}$. In addition, if in the last iteration the columns of A become the columns of I, then $\mathbf{B}^{-1} = \mathbf{A}^{-1}$.

If the optimum (final) simplex table does not contain all the variables of vector $\mathbf{X}$ in the basis, we continue simplex procedure until all the variables of vector $\mathbf{X}$ are in the basis and at the same time the solution remains optimum, may be feasible or infeasible.

Note. The dummy vector **b** can be constructed easily by assigning the value one to all the variables of vector **X**.

Following example will illustrate the procedure :

Example 30. *Use simplex method to obtain the inverse of the matrix* $A = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$.

Solution. Consider the matrix equation

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}; x_1, x_2 \geq 0,$$

where the right side $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is a *dummy vector.* Now apply simplex method to maximize the dummy objective function $z = 0x_1 + 0x_2 - 1x_{3a} - 1x_{4a}$, subject to the above constraints on x_1, x_2, where $x_{3a} \geq 0$ and $x_{4a} \geq 0$ are artificial variables.

Table 5.70

		$c_j \rightarrow$	0	0	−1	−1	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_{3A}	X_{4A}	MIN. RATIO (X_B/X_K)
x_{3a}	−1	4	←[3] ↑	2	1	0	4/3 ←
x_{4a}	−1	6	4	−1	0	1	6/4
	$z = -10$		−7 ↑	−1	0 ↓	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	0	4/3	1	2/3	1/3	0	
$\leftarrow x_{4a}$	−1	2/3	0	[−11/3]	−4/3	1	
	$z = -2/3$		0	11/3 ↑	7/3	0 ↓	$\leftarrow \Delta_j \geq 0$
x_1	0	16/11	1	0	1/11	2/11	
x_2	0	−2/11	0	1	4/11	−3/11	
	$z = 0$		0	0	1	1	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, an optimum solution is obtained. But matrix A is not yet converted into unit matrix. To do so, introduce $\mathbf{X_2}$ in the basis and remove $\mathbf{X_{4A}}$ from the basis.

Now an optimum (but infeasible) solution has been obtained. Since the initial basis consisted of column vectors $\mathbf{X_{3A}}$ and $\mathbf{X_{4A}}$, the inverse of matrix A is given by

$$\begin{bmatrix} 1/11 & 2/11 \\ 4/11 & -3/11 \end{bmatrix}.$$

Example 31. *Following is the final optimal table for a given L.P. problem, answer that it originally has an identity matrix under* $\mathbf{X_3}$ and $\mathbf{X_4}$.

(a) *What is the value of the objective function for optimal solution ?*

(b) *What is the optimal basis ? Give a* 2×2 *numerical matrix.*

(c) *Are there any alternative optimal solutions ? If so, which variable gives an alternative optimal solution ?*

Table 5·71

		$c_j \rightarrow$	2	2	0	0
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4
x_1	2	5	1	0	1/2	– 1/2
x_2	2	4	0	1	– 1/2	3/2
			0	0	0	2

(d) Suppose c_2 was equal to 3 instead of 2. Would we have still the optimal solution.

(e) Without calculating write the inverse of the optimal basis found in (b) above.

Solution. (a) The optimal solution is given as $x_1 = 5$, $x_2 = 4$.

The value of objective function will be

$$z = \mathbf{C_B X_B} = (2 , 2) (5 , 4) = 2 \times 5 + 2 \times 4 = 18 .$$

(b) Find the optimal basis **B** as follows :

Since $\mathbf{B}^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$, and $\mathbf{BB}^{-1} = \mathbf{I}$, we have $\mathbf{B}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now apply the elementary row operations : $R_1 \rightarrow 2R_1$ and $R_2 \rightarrow 2R_2$ and we get

$$\mathbf{B} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Again applying $R_2 \rightarrow R_2 + R_1$, $\mathbf{B}.\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$.

Further applying $R_2 \rightarrow (1/2)\ R_2$, $\mathbf{B}.\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Again applying $R_1 \rightarrow R_1 + R_2$, $\mathbf{B}.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ or $\mathbf{BI} = \mathbf{B} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. **Ans.**

(c) Since $\Delta_3 = 0$ and the variable x_3 is not in the basis, it shows that alternative optimum solutions exist. Therefore, if $\mathbf{x}_3$ enters the basis and $\mathbf{x}_1$ leaves the basis, an alternative optimum simplex table is obtained as follows :

Table 5·72

		$c_j \rightarrow$	2	2	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	
x_3	0	10	2	0	1	–1	
x_2	2	9	1	1	.0	1	
		$z = 18$	0	0	0	2	$\leftarrow \Delta_j$

Thus optimum solution is : $x_1 = 0$, $x_2 = 9$.

(d) If c_2 becomes equal to 3 instead of 2, then there exists a unique optimal solution $x_1 = 0$, $x_2 = 9$.

(e) Inverse of optimal basis from the optimal table is read as

$$\mathbf{B}^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

EXAMINATION PROBLEMS

1. Solve the following system of linear equations by simplex method.

(i) $x_1 + x_2 = 1$
$2x_1 + x_2 = 3$. **[Meerut M.Sc. (Math.) 90]**

(ii) $3x_1 + 2x_2 = 4$
$4x_1 - x_2 = 6$ **[Meerut M.Sc. (Math.) 94]**

[Ans. $x_1 = 2$, $x_2 = -1$]

2. Find the inverse of the matrix

(i) $\begin{bmatrix} 4 & 1 & 2 \\ 0 & 1 & 0 \\ 8 & 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 1 \\ 2 & 9 \end{bmatrix}$

[Meerut (M.Sc.) 93]

3. Consider the matrix $\mathbf{B} = (\beta_1, \beta_2, \beta_3)$ whose inverse is

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

Find the inverse matrix $\mathbf{B} = (\beta_1, \beta_2, \mathbf{e})$ where $\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

3. Solve the system of equations by using simplex method :

$3x_1 + 2x_2 + x_3 + 4x_4 \le 6$, $2x_1 + x_2 + 5x_3 + x_4 \le 4$, $2x_1 + 6x_2 - 4x_3 + 8x_4 = 0$,

$x_1 + 3x_2 - 2x_3 + 4x_4 = 0$, and $x_1\ x_2$, $x_3\ x_4 \ge 0$. **[Meerut (Maths.) Jan. 98 (BP)]**

4. (a) In relation to linear programming, explain the implications of the following assumptions of the model :
(i) Linearity of the objective function and constraints ; (ii) Continuous variables; (iii) Certainty.

(b) An Air Force is experimenting with three types of bombs P, Q and R in which three kinds of explosives, viz. A, B and C will be used. Taking the various factors into consideration, it has been decided to use at most 600 kg of explosive A, at least 480 kg of explosive B and exactly 540 kg of explosive C. Bomb P requires 3, 2, 2 kg of A, B and C respectively. Bomb Q requires 4, 3, 2 kg. of A, B and C. Bomb R requires 6, 2, 3 kg of A, B and C respectively. Now bomb P will give the equivalent of a 2-ton explosion, bomb Q will give a 3-ton explosion and bomb R will give a 4-ton explosion. Under what production schedule can the Air Force make the biggest bomb.

(c) Obtain the dual problem of the following L.P.P. : Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$, subject to the constraints :
$5x_1 + 6x_2 - x_3 \le 3$, $-x_1 + x_2 + 3x_3 \ge 4$, $7x_1 - 2x_2 - x_3 \le 10$, $x_1 - 2x_2 + 5x_3 \ge 3$, $4x_1 + 7x_2 - 2x_3 \ge 2$, and x_1 , x_2 , $x_3 \ge 0$

[ICWA (June) 91]

5·14. SUMMARY OF COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

Simplex method is an iterative procedure involving the following steps :

Step 1. If the problem is one of minimization, convert it to a maximization problem by considering $-z$, instead of z, using the fact $\min z = -\max(-z)$ or $\min z = -\max.(z')$, $z' = -z$.

Step 2. We check up all b_i 's for nonnegativity. If some of the b_i 's are negative, multiply the corresponding constraints through by -1 in order to ensure all $b_i \ge 0$.

Step 3. We change the inequalities to equations by adding slack and surplus variables, if necessary.

Step 4. We add artificial variables to those constraints with ($\ge$) or (=) sign in order to get the identity basis matrix.

Step 5. We now construct the starting simplex table (*see Table 5·73* for all problems). From this table, the initial basic feasible solution can be read off.

Table 5·73. Form of Simplex Table

		$c_j \rightarrow$	c_1	c_2	$c_3 \dots c_k \dots c_{m+n}$	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3} \dots \mathbf{X_k} \dots \mathbf{X_{m+n}}$	MIN. RATIO RULE
...	...	...	...	...	... ...	...
	$z = \mathbf{C_B X_B}$		Δ_1	Δ_2	$\Delta_3 \dots \Delta_k \dots \Delta_{m+n}$	$\leftarrow \Delta_j$

Note. All the steps of simplex algorithm can be easily remembered by the **Flow-Chart** given below

Step 6. We obtain the values of Δ_j by the formula $\Delta_j = z_j - c_j = \mathbf{C_B X}_j - c_j$, and examine the values of Δ_j . There will be three mutually exclusive and collectively exhaustive possibilities :

(i) All $\Delta_j \ge 0$. In this case, the basic feasible solution under test will be optimal.

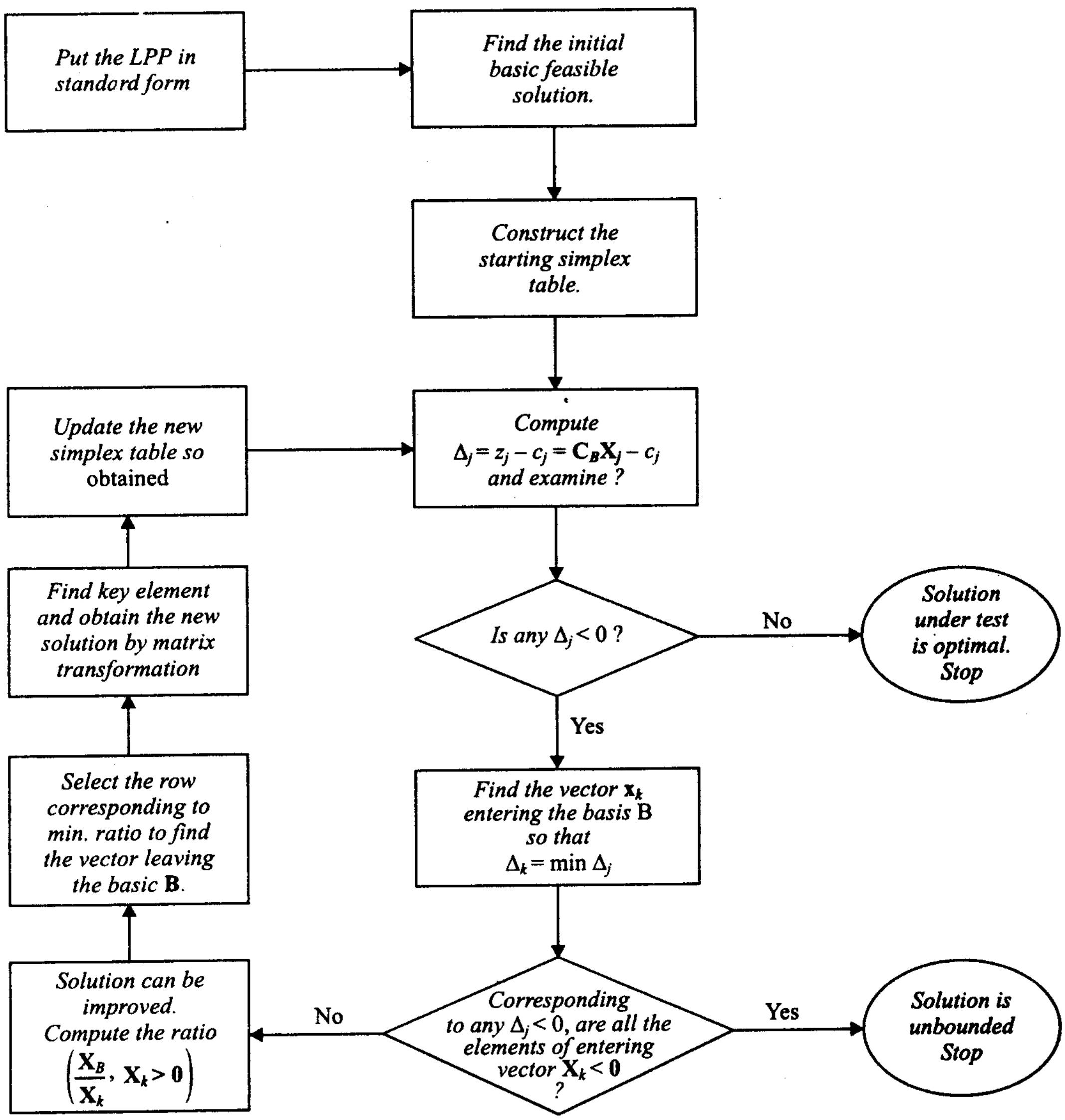
(ii) Some $\Delta_j < 0$, and for at least one of the corresponding $\mathbf{x}_j$ all $x_{rj} \le 0$. In this case, the solution will be unbounded.

(iii) Some $\Delta_j \le 0$, and all the corresponding $\mathbf{x}_j$'s have at least one $x_{ij} > 0$. In this case, there is no end of the road. So further improvement is possible.

Step 7. Further improvement is done by replacing one of the vectors at present in the basis matrix by that one out side the basis. We use the following rules to select such a vector :

(i) *To select "incoming vector". We find such value of k* for which $\Delta_k = \min \Delta_j$. Then the vector coming into the basis matrix will be $\mathbf{x_k}$.

FLOWCHART FOR SIMPLEX METHOD



(ii) *To Select "outgoing vector". The vector going out of the basis matrix will be* β_r, *if we determine the suffix r* by the minimum ratio rule

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0\right], \qquad \text{for predetermined value of } k.$$

Step 8. We now construct the next improvement table by using the simple matrix transformation rules.

Step 9. Now return to ***step 6,*** then go the ***steps 8*** and ***9,*** if necessary. This process is repeated till we reach the desired conclusion.

Q. 1. Give the outlines of simplex method of linear programming. **[JNTU (MCA III) 2004; Banasthali (M.Sc.) 93]**

2. Show how simplex method can be applied to find a solution of the following system : $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq 0$

3. Give a flow chart of the simplex method. **[Kanpur (B.Sc.) 91]**

SELF EXAMINATION PROBLEMS

1. Solve the following problems by simplex method adding aftificial variables :

(a) Min. $z = 2x_1 - 3x_2 + 6x_3$, subject to

$3x_1 - 4x_2 - 6x_3 \leq 2$
$2x_1 + x_2 + 2x_3 \geq 11$
$x_1 + 3x_2 - 2x_3 \leq 5$
$x_1, x_2, x_3 \geq 0$.

(b) Max. $z = 2x_1 + 5x_2 + 7x_3$, subject to

$3x_1 + 2x_2 + 4x_3 \leq 100$
$x_1 + 4x_2 + 2x_3 \leq 100$
$x_1 + x_2 + 3x_3 \leq 100$
$x_1, x_2, x_3 \geq 0$

[Ans. $x_1 = 0$, $x_2 = 50/3$, $x_3 = 50/3$, $max z = 200$]

(c) Maximize $z = x_1 + 3x_2 + 5x_3 + 4x_4 + 5x_5$

subject to $3x_1 + 2x_2 + 4x_3 + x_4 + 5x_5 = 15$
$x_1 + 2x_2 + x_3 + 5x_4 + 5x_5 = 13$
$2x_3 + 6x_4 + 3x_5 \geq 6$
$x_1, x_2, x_3, x_4, x_5 \geq 0$.

[Meerut (L.P.) 90]

2. Max. $z = 4x_1 + x_2 + 4x_3 + 5x_4$, subject to the constraints :

$4x_1 + 6x_2 - 5x_3 + 4x_4 \geq -20$, $3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$, $8x_1 - 3x_2 - 3x_3 + 2x_4 \leq 20$, and $x_1, x_2, x_3, x_4 \geq 0$.

[Ans. Unbounded solution]. **[Agra 98]**

3. A manufacturer has two products P_1 and P_2 , both of them are produced in two steps by machines M_1 and M_2 . The process times per hundred for the products on the machines are :

	M_1	M_2	Contribution (per hundred)
P_1	4	5	10
P_2	5	2	5
Available (hrs)	100	80	

The manufacturer is in a market upswing and can sell as much as he can produce of both products. Formulate the mathematical model and determine optimum product mix using simplex method.

[Hint. Formulation of the problem is : Max. $z = 10x_1 + 5x_2$, subject to $4x_1 + 5x_2 \leq 100$, $5x_1 + 2x_2 \leq 80$, $x_1, x_2 \geq 0$] .

[Ans. $x_1 = 20000/17 \approx 1177$, $x_2 = 18000/17 \approx 1059$].

4. An animal feed company must produce 200 kgs of a mixture consisting of ingredient X_1 and X_2 daily. X_1 costs Rs. 3 per kg and X_2 Rs. 8 per kg. No more than 80 kgs of X_1 can be used and at least 60 kgs of X_2 must be used.Find how much of each ingredient should be used if the company wants to minimize cost.

[Hint. Formulation of the problem is : Min. $3x_1 + 8x_2$, subject to $x_1 + x_2 = 200$, $x_1 \leq 80$, $x_2 \geq 60$, $x_1, x_2 \geq 0$,

Substitute $x_1 = X_1 + 80$ in the problem and then solve by simplex method.

[Ans. $x_1 = 80$, $x_2 = 120$, min cost = Rs. 1200].

5. A manufacture of leather belts makes three types of belts A , B and C which are processed on three machines M_1 , M_2 and M_3 . Belt A required 2 hours on machine M_1 , and 3 hours on machine M_3 . Belt B requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3 ; and Belt C required 5 hours on machine M_2 and 4 hours on machine M_3 . There are 8 hours of time per day available on machine M_1 ,10 hours per day avaiable on machine M_2 and 15 hours of time per day available on machine M_3 . The profit gained from Belt A is Rs. 3 per unit, from Belt B is Rs. 5 per unit, from Belt C is Rs. 4 per unit, what should be the daily production of each type of belts so that the profit is maximum. **[Agra 2000; Kanpur (B.A) 91]**

6. A company produces three products A , B and C . These products require three ores O_1 , O_2 and O_3. The maximum quantities of the ores O_1 , O_2 and O_3 available are 22 tons, 14 tons and 14 tons respectively. For one ton of each of these products, the ore requirements are :

	A	B	C
O_1	3	—	3
O_2	1	2	3
O_3	3	2	0
Profit per ton (Rs. in thousand)	1	4	5

The company makes a profit of one, four and five thousands on each ton of the products A , B and C respectively. How many tons of each products A , B and C should the company produce to maximize the profits.

[Hint. Formulation of the problem is : Max. $z = x_1 + 4x_2 + 5x_3$,

subject to $3x_1 + 3x_3 \leq 22$, $x_1 + 2x_2 + 3x_3 \leq 14$; $3x_1 + 2x_2 \leq 14$; $x_1, x_2, x_3 \geq 0$.]

7. A furniture company manufactures four models of desks. Each desk is first constructed in the carpentry shop and is next sent to the finishing shop where it is varnished, waxed and polished. The number of man-hours of labour required in each shop is as follows :

Shop	Desk			
	I	II	III	IV
Carpentry	4	9	7	10
Finishing	1	1	3	40
Profit per item (Rs.)	12	20	18	40

Because of limitation in capacity of the plant, not more than 6,000 man-hours can be expected in the carpentry shop and 4,000 in the finishing shop in a month. Assuming that raw materials are available in adequate supply and all desks produced can be sold, determine the quantities of each type of desk to be made for maximum profit of the company.

[Hint. Formulation of the problem is : Max. $z = 12x_1 + 20x_2 + 18x_3 + 40x_4$,

subject to $4x_1 + 9x_2 + 7x_3 + 10x_4 \leq 6{,}000$, $x_1 + x_2 + 3x_3 + 40x_4 \leq 400$; $x_1, x_2, x_3, x_4 \geq 0$.

[Ans. $x_1 = 4000/3$, $x_2 = x_3 = 0$, $x_4 = 200/3$, and max. $z = Rs.\ 56000/3$]

8. A factory has decided to diversify their activities, and data collected by sales and production is summerized below :

Potential demand exists for 3 products A , B and C . Market can take any amount of A and C whearas the share of B for this organization is expected to be not more than 400 units a month.

For every three units of C produced, there will be one unit of a by-product which sells at a contribution of Rs. 3 per unit, and only 100 units of this by-product can be sold per month. Contribution per unit of products A , B and C is expected to be Rs. 6, Rs. 8 and Rs. 4 respectively.

These products require 3 different processes, and time required per unit production is given in the following table :

Process	Product (hrs/unit)			Available
	A	B	C	
I	2	3	1	900
II	—	1	2	600
III	3	2	2	1200

Determine the optimum product-mix for maximizing the contribution.

[Hint. The formulation of the problem is : Max. $z = 6x_1 + 8x_2 + 4x_3 + 3x_4$, subject to

$2x_1 + 3x_2 + x_3 \leq 900$,$3x_1 + 2x_2 + 2x_3 \leq 1200$, $3x_3 + x_4 = 100$; $x_2, x_3 \geq 0$],

[Ans. $x_1 = 360$, $x_2 = 60$, $x_3 = 0$, $x_4 = 100$; max z = Rs. 2940]

9. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yields a profit of Rs. 30. An acre of wheat costs Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans costs Rs. 70 to prepare requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8000 may-days of work, how many acres should be allocated to each crop to maximize profit ?

[Hint. Formulation of the problem is : Max $z = 30x_1 + 40x_2 + 20x_3$, subject to

$10x_1 + 12x_2 + 7x_3 \leq 10{,}000$, $7x_1 + 10x_2 + 8x_3 \leq 8000$, $x_1 + x_2 + x_3 \leq 1000$, and $x, x_2, x_3 \geq 0$.

[Ans. $x_1 = 250$, $x_2 = 625$, $x_3 = 0$; max z = Rs. 32,500].

10. A transistor radio company manufactures four models A , B , C and D which have profit contributions of Rs. 8, Rs. 15 and Rs. 25 on models A , B and C respectively and a loss of Re. 1 on model D . Each type of radio requires a certain amount of time for the manufacturing of components for assembling and for packing. Specially a dozen units of model A require one hour of manfacturing, two hours for assembling and one hour for packing. The corresponding figures for a dozen units of model B are 2, 1 and 2 and for a dozen units of C are 3, 5 and 1, while a dozen units of model D require 1 hour of packing only, During the forthcoming week, the comapny will be able to make available 15 hours of manufacturing, 20 hours of assembling and 10 hours of packing time. Obtain the optimal production schedule for the company.

[Hint. Formulation of the problem is : max. $z = 8x_1 + 15x_2 + 25x_3 - x_4$, subject to

$x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$; $x_1, x_2, x_3, x_4 \geq 0$.

[Ans. $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$, max z = Rs. 120].

11. A manufacturing firm has discountinued production of a certain unprofitable product line. This created considerable excess production capacity. Management is considering to devote this excess capacity to one or more of three products : call them product 1, 2 and 3. The available capacity on the machines which might limit output, is summarized in the following table.

Machine Type	Available Time (in machine hours per week)
Milling Machine	250
Lathe	150
Grinder	50

The number of machine-hours required for the unit of the respective product is given below :

Machine Type	Productivity 33 (in machine-hours per unit)		
	Product 1	Product 2	Product 3
Milling Machine	8	2	3
Lathe	4	3	0
Grinder	2	--	1

The unit profit would be Rs. 20, Rs. 6 and Rs. 8 respectively for products 1, 2 and 3. Find how much of each product the firm should produce in order to maximize profit ?

[**Hint.** Formulation of the problem is : Max. $z = 20x_1 + 6x_2 + 8x_3$, subject to
$8x_1 + 2x_2 + 3x_3 \le 250$; $4x_1 + 3x_2 \le 150$, $2x_1 + x_3 \le 50$, and x_1 , x_2 , $x_3 \ge 0$.

[**Ans.** $x_1 = 0$, $x_2 = 50$, $x_3 = 50$; max $z = 700$].

12. The *XYZ* company manufactures two products *A* and *B* . These products are processed on the same machine. It takes 25 minutes to process one unit of product *A* and 15 minutes for each unit of product *B* and the machine operates for a maximum of 35 hours in a week. product *A* requires 1 kg. of the raw material per unit, the supply of which is 170 kgs. per week.

If the net income from the products are Rs. 100 and Rs. 450 per unit respectively and manufacturing costs are proportional to the square of the quantity made for each product, find how much of each product should be produced per week, in order to maximize profits.

[**Ans.** 68 units of *B* , no unit of *A* , max profit = Rs. 30,600].

13. A manufacturer of steel furniture makes three products—Chairs, Filing cabinetes and Tables. Three machines (call them *A* , *B* and *C*) are available on which these products are processed. The manufacturer has 100 hours per week available on each of three machines. The time required by each of the three products on three machines is summarized in the following table :

Product	Time required (in hours)		
	Machine *A*	Machine *B*	Machine *C*
Chair	2	2	1
Filing cabinet	2	1	2
Table	—	1	2

The profit analysis shows that the net profit on each chair, filing–cabinet and table is Rs. 22, Rs. 30 and Rs. 25 respectively.

What should be the weekly production of these products so that the manufacturer's total profit per week is maximized.

[**Ans.** 100/3 chairs, 50/3 tables, and 50/3 file cabintes. max. profit = Rs. 1650]

14. A manufacture produces three products *A* , *B* and *C*. Each product can be produced on either one of two mac[illegible] and II. The time required to produce 1 unit of each product on a machine is given in the table below :

Product	Time to Produce 1 unit (hours)	
	Machine I	Machine II
A	0·5	0·6
B	0·7	0·8
C	0·9	1·05

There are 85 hours available on each machine ; the operating cost is Rs. 5 per hour for machine I and Rs. 4 per hour for machine II and the product requirements are at least 90 units of *A* , at least 80 units *B* , and at least 60 units of *C*. The manufacturer wishes to meet the requirements at minimum cost.

Solve the given linear programming problem by simplex method.

[**Ans.** 150 hrs. on machine II, no time on machine I, min cost = Rs. 600]

15. A Plant is engaged on the production of two products which are processed through three departments, the number of hours required to finish each is indicated in the table below :

Department	Product		Max. hours available per week
	A	*B*	
I	7	8	1600
II	8	12	1600
III	15	16	1600

(a) If the profit for the products is Rs. 6 for a unit of product *A* but only Rs. 4 for a unit of product. *B* , what quantities per week should be planned to maximize profit. Illustrate the problem graphically.

(b) Capacity can be increased on one department only. In which department should it be done and why ? To what extent should the capacity be increased ?

(c) If the cost per hour in department I is Rs. 25 ; in department II, Rs. 40; and in department III, Rs. 50 ; what quantities should be planned to minimize the cost of production ?

[**Ans.** (a) 320/3 units of product *A* and no unit of product *B* ; max. profit is Rs. 640.
(c) 10/3 units of product *A* and no unit of product *B* ; min. cost = Rs. 20].

16. A factory is engaged in manufacturing three products A, B and C which involve lathe work, grinding and assembling. The cutting, grinding and assembling time required for one unit of A are 2, 1 and 1 hours respectively. Similarly they are 3, 2 and 3 hours for one unit of B and 1, 3 and 1 hours for one unit of C. The profits on A, B and C are Rs. 2, Rs. 2 and Rs. 4 per unit, respectively. Assuming that there are available 300 hours of lathe time, 300 hours of grinder time and 240 hours of assembly time; how many units of each product should be produced to maximize profits ? Work the problem using the simplex method.

17. A teacher gives his students three long lists of problems with the instruction to submit not more than 100 of them correctly solved, for credit. The problems in the first list are of 5 points each, in second 4 points each, and in third 6 points each. On an average 3 minutes are required to solve a problem from first list, 2 minutes for a problem from second and 4 minutes for a problem from third. The students devote more than $3\frac{1}{2}$ hours for mathematics. The first two lists of problems involve numerical calculations and the students cannot do more than $2\frac{1}{2}$ hours of numerical work. How many problems from each list a student should solve so as to get the maximum credit.

18. Two products A and B are processed on three machines M_1, M_2, M_3. The processing times per unit, machine availability and profit per unit are as under :

Machine	Processing time (hours)		Availability (Hours)
	A	B	
M_1	2	3	1500
M_2	3	2	1500
M_3	1	1	1000
Profit per unit	10	12	

Formulate the mathematical model, solve it by using simplex technique and also find the number of hours machine M_3 remains unutilized.

19. Formulate and solve the following linear programming problem using the simplex procedure.

A manufacturer makes straight chairs and rotating chairs, For each type of chair he used three major production areas of cutting, dipping and assembly. Capacities of the three areas during the next week are :

Cutting : 200 straight, or 300 rotating or any combination.
Dipping : 400 straight or 400 rotating or any combination.
Assembly : 250 straight or 200 rotating or any combination.

From a straight chair he makes a profit of Rs. 5 and from a rotating chair a profit of Rs. 10. Determine the number of straight chairs and number of rotating chairs he must manufacture so as to have maximum profit.

20. The postmaster of a local post office wishes to hire extra helpers during the Deepawali season, because of a large increase in the volume of mail handling and delivery. Bacause of the limited office space and budgetary condition, the number of temporary helpers must not exceed 10. According to the past experience, men can handle 300 letters and 80 packages per day, on the average, and women can handle 400 letters and 50 packages per day. The postmaster believes that the daily volume of extra mail and packages will be no less than 3,400 and 680 respectively. A man receives Rs. 25 a day and a woman receives Rs. 22 a day. How many man and woman helpers should he hired to keep the pay-roll at a minimum ?

21. A television comapny has three major departments for manufacture of its models A and B. Monthly capacities are given as follows :

Department	Per Unit Time Requirement (Hrs.)		Hrs. available this month
	Model A	Model B	
I	4·0	2·0	1,600
II	2·5	1·0	1,200
III	4·5	1·5	1,600

The marginal prifit of model A is Rs. 400 each and that of model B is Rs. 100 each. Assuming that the company can sell any quantity of either product due to favourable market conditions; determine the optimum output for both the models, the highest possible profit for this month and the slack time in the three departments.

[**Hint.** Formulation is : Max. $z = 400x_1 + 100x_2$, subject to the conditions :

$$4x_1 + 2x_2 \le 1600;\ 5/2\ x_1 + x_2 \le 1200\ ;\ 9/2\ x_1 + 3/2\ x_2 \le 1600\ ;\ x_1 \ge 0\ ,\ x_2 \ge 0]$$

[**Ans.** $x_1 = 3200/9$, $x_2 = 0$, $x_3 = 0$, max. $z = 1280000/9$]

22. Consider the linear programming problem : Max. $4x + 10y$ subject to the constraints :

$$2x + y \le 50\ ,\ 2x + 5y \le 100\ ,\ 2x + 3y \le 90\ ,\ \text{and } x\ ,\ y \ge 0\ .$$

(a) Solve this problem graphically.
(b) Solve this problem by the simplex method. Is the solution unique ? why or why not ? If not, give two different basic optimal solutions.
(c) Find all the optimal solutions (not necessarily basic) to the problem.

23. Solve L.P. Problem :

$$\begin{aligned} \text{Maximize} \quad & z = 2x_1 - 2x_2 + 3x_3 \\ \text{Subject to} \quad & 2x_1 + 3x_2 - x_3 \le 30 \\ & 3x_1 - 2x_2 + x_3 \le 24 \\ & x_1 - 4x_2 - 6x_3 \ge 2 \\ & x_1\ ,\ x_2\ ,\ x_3 \ge 0. \end{aligned}$$

[JNTU (B. Tech.) 97]

24. A small paint factory produces three types of paints as follows :

Paint	Production (Kg./day)	Profit (Units/Kg)
1	x	10
2	y	4
3	z	1

25. A firm manufactures three products which are processed on three different machines. The relevant data are as follows :

Time per unit (hrs.)

Machine	Product I	Product II	Product III
M_1	2	3	2
M_2	4	—	3
M_3	2	5	—

The machine capacities for M_1, M_2 and M_3 are respectively 440, 470 and 430 hrs while the unit profits for PI, PII and PIII are Rs. 400, 300 and 600 respectively. Assume that all the products are sold.

(i) Formulate the problem as an LPP.

(ii) Express the LPP in standard form.

(iii) Find an initial basic feasible solution.

(iv) Carry out up to three iterations towards optimal solution using simplex algorithm. Clearly explain our key steps.

[AIMS Bang. (MBA) 2002]

26. Solve the following LPP using Big-M method. Minimize $z = 2x_1 + 5x_2$, subject to $x_1 + x_2 = 100$, $x_1 \le 40$, $x_2 \le 30$, $x_1, x_2 \ge 0$. **[VTU 2003]**

SELF–EXAMINATION QUESTIONS

1. Establish the difference between (i) feasible solution, (ii) Basic feasible solution and (iii) degenerate basic feasible solution.
2. (a) Define a basic solution to a given system of m simultaneous linear equations in n unknowns.
   (b) How many basic feasible solutions are there to a given system of 3 simultaneous linear equations in 4 unknown.
3. Define the following terms :
   (i) basic variable (ii) basic solution (iii) basic feasible solution (iv) degenerate solution.
4. Give outlines of simplex method in linear programming. Why is it so called.
5. What do you mean by two phase method for solving a given L.P.P. Why is it used.
6. What are the various methods known to you for solving a linear programming problem ?
7. What is the pivoting process ?
8. Name the three basic parts of the simplex technique.
9. Give the geometric interpretation of the simplex procedure.
10. Write the role of pivot element in a simplex table. **[Madurai B.Sc. (Com. Sc.) 92]**
11. In the course of simplex table calculations, describe how you will detect a degenerate, an unbounded and a non existing feasible solution.
12. What is degeneracy in simplex ? Solve the following LP problem using simplex :
    Max. $z = 3x_1 + 9x_2$, s.t. $4x_1 + 4x_2 \le 8$, $x_1 + 2x_2 \le 4$ and x_1, $x_2 \ge 0$. **[IPM (PGDBM) 2000]**
13. With reference to the solution of LPP by simplex method/table when do you conclude as follows :
    (i) LPP has multiple solutions, (ii) LPP has no limit for the improvement of the objective function, (iii) LPP has no feasible solution. **[VTU 2002]**

OBJECTIVE QUESTIONS

1. The role of artificial variables in the simplex method is
   (a) to aid in finding an initial solution. (b) to find optimal dual prices in the final simplex table.
   (c) to start phases of simplex method. (d) all of the above.
2. For a maximization problem, the objective function coefficient for an artificial variable is
   (a) + M. (b) – M. (c) zero. (d) none of the above.
3. If a negative value appears in the solution values (x_B) column of the simplex table, then
   (a) the solution is optimal. (b) the solution is infeasible.
   (c) the solution is unbounded. (d) all of the above.
4. At every iteration of simplex method, for minimization problem, a variable in the current basis is replaced with another variable that has
   (a) a negative $z_j - c_j$ value. (b) a positive $z_j - c_j$ value.
   (c) $z_j - c_j = 0$. (d) none of the above.

5. In the optimal simplex table, $z_j - c_j = 0$ indicates
(a) unbounded solution. (b) cycling. (c) alternative solution. (d) infeasible solution.

6. For maximization LP model, the simplex method is terminated when all values
(a) $z_j - c_j \geq 0$. (b) $z_j - c_j \leq 0$. (c) $z_j - c_j = 0$. (d) $z_j \leq 0$.

7. A variable which does not appear in the basic variable (*B*) column of simplex table is
(a) never equal to zero. (b) always equal to zero. (c) called a basic variable. (d) none of the above.

8. If for a given solution, a slack variable is equal to zero, then
(a) the solution is optimal.
(b) the solution is infeasible.
(c) the entire amount of resource with the constraint in which the slack variable appears has been consumed.
(d) all of the above.

9. If an optimal solution is degenerate, then
(a) there are alternative optimal solutions (b) the solution is infeasible
(c) the solution is of no use to the decision-maker (d) none of the above.

10. To formulate a problem for solution by the simplex method, we must add artificial variable to
(a) only equality constraints (b) only 'greater than' constraints
(c) both (a) and (b) (d) none of the above.

11. Lower bound constraints are handled by substituting :
(a) $x_j = u_j - x_j$ (b) $x_j = 4j + x_j$ (c) $x_j = l_j - x_j$ (d) $x_j = l_j - x_j$

12. Upper bound constraints are handled by substituting
(a) $x_j = u_j - x_j''$ (b) $x_j = u_j + x_j''$ (c) $x_j = l_j - x_j''$ (d) $x_j = l_j + x_j''$

13. When a non-basic variable is at its upper bound, the remaning non-basic variables are put at zero value by using the reletionship.
(a) $x_r = u_r + x_r'$ (b) $x_r = u_s + x_r'$ (c) $x_r = l_r + x_r'$ (d) $x_r = l_r + x_r'$

14. Lower and upper bounds in case of an unbounded variable is
(a) 0 and ∞ (b) $-\infty$ and ∞ (c) 0 and $-\infty$ (d) none of these

Answers

1. (d) **2.** (a) **3.** (d) **4.** (a) **5.** (d) **6.** (c) **7.** (d) **8.** (a) **9.** (a) **10.** (c) **11.** (d) **12.** (a) **13.** (b) **14.** (a).

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6

REVISED SIMPLEX METHOD

6.1. INTRODUCTION

The usual simplex method described so far is a straight forward algebraic procedure. But the examination of the sequence of calculations in the usual simplex method, however, leads to the following disadvantages :

(i) It is very time-consuming even when considered on the time scale of electronic digital computers. Hence it is not an efficient computational procedure.

(ii) In the usual simplex method, many numbers are computed and stored which are either never needed at the current iteration or are needed only in an indirect way.

(iii) It does not give the inverse and simplex multipliers. Although it is possible to modify the ordinary simplex method to give the inverse and simplex multipliers, but this would in general increase the computational effort.

Keeping this in view, a *revised simplex method* has been developed to overcome these disadvantages, which consequently speed up the calculations by reducing the required amount of computational effort. In general, approach of the revised simplex method is identical to that on which the ordinary simplex method is based.

Proceeding from one iteration to the other in the simplex method, it was unnecessary to transform all the $\mathbf{X}_j$, $\mathbf{X}_\mathbf{B}$, $z_j - c_j$ and z at each iteration. If fact, all new quantities ($\mathbf{B}^{-1}$, $\mathbf{X}_\mathbf{B}$, $\mathbf{C}_\mathbf{B}\,\mathbf{B}^{-1}$, z) can be computed directly from their definitions, provided $\mathbf{B}^{-1}$ is known ; that is if, only the basis inverse is transformed and only such $\mathbf{X}_j$ is determined at each iteration for which the vector is entered in basis. Thus only the parts of information relevent at each iteration are :

(i) coefficients of non-basic variables in the objective function $z = \mathbf{CX}$;

(ii) coefficient of the entering basic variable in the system of constraint equations $\mathbf{AX} = \mathbf{b}$; and

(iii) right side of the equation $\mathbf{AX} = \mathbf{b}$, that is, the vector $\mathbf{b}$.

6.2. STANDARD FORMS FOR REVISED SIMPLEX METHOD

There are two *standard forms* for the revised simplex method :

Standard Form I. In this form, it is assumed that an identity (basis) matrix is obtained after introducing slack variables only.

Standard Form II. If artificial variables are needed for an initial identity (basis) matrix, then *two-phase method* of ordinary simplex method is used in a slightly different way to handle artificial variables.

The *revised simplex method* is now discussed in above two standard forms separately.

Revised Simplex Method in Standard Form–I

6.3. FORMULATION OF LP PROBLEM IN STANDARD FORM-I

A linear programming problem in standard form is :

$$\text{Max. } z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \dots 0x_{n+m}, \text{ subject to} \quad \dots(6\cdot1)$$

$$\left.\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} & & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & + x_{n+2} & & = b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & & + x_{n+m} & = b_m \end{array}\right\} \quad \dots(6\cdot2)$$

and

$$x_1, x_2, \dots, x_{n+m} \geq 0, \quad \dots(6\cdot3)$$

where the starting basis matrix $\mathbf{B}$ is an $m \times m$ identity matrix.

In the revised simplex form, the objective function (6·1) is also considered as if it were another constraint in which z is as large as possible and unrestricted in sign.

Thus, (6·1) and (6·2) may be written in a compact form as:

$$\left.\begin{array}{llllllll} z - c_1x_1 & - c_2x_2 - \ldots - & c_nx_n + 0x_{n+1} + 0\,x_{n+2} + \ldots + 0x_{n+m} & & = 0 \\ a_{11}x_1 & + a_{12}x_2 + \ldots + & a_{1n}x_n & + x_{n+1} & = b_1 \\ a_{21}x_1 & + a_{22}x_2 + \ldots + & a_{2n}x_n & + x_{n+2} & = b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 + \ldots + & a_{mn}x_n & + x_{n+m} & = b_m \end{array}\right\} \quad \ldots(6{\cdot}4)$$

which can be considered as a system of $m+1$ simultaneous equations in $(n+m+1)$ number of variables $(z, x_1, x_2, \ldots, x_{n+m})$. Here our aim is to find the solution of the system (6·4) such that z is as large as possible and unrestricted in sign.

Now, the system (6·4) may be re-written as follows :

$$\left.\begin{array}{l} 1.x_0 + a_{01}x_1 + a_{02}x_2 + \ldots + a_{0n}x_n + a_{0,n+1}x_{n+1} + \ldots + a_{0,n+m}x_{n+m} = 0 \\ 0.x_0 + a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + 1.x_{n+1} + \ldots + 0.x_{n+m} = b_1 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ 0.x_0 + a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + 0x_{n+1} + \ldots + 1.x_{n+m} = b_m \end{array}\right\} \quad \ldots(6{\cdot}5)$$

where $z = x_0$ and $-c_j = a_{0j}$ $(j = 1, 2, \ldots, n+m)$.

Again, writing the system (6·5) in matrix form,

$$\begin{bmatrix} 1 & : & a_{01} & a_{02} \ldots a_{0n} & a_{0,n+1} \ldots a_{0,n+m} \\ \ldots\ldots : & \ldots & \ldots & \ldots & \ldots \\ 0 & : & a_{11} & a_{12} \ldots a_{1n} & 1 \quad\quad 0 \\ \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots \quad\quad \vdots \\ 0 & : & a_{m1} & a_{m2} \ldots a_{mn} & 0 \quad\quad 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \quad \ldots(6{\cdot}6)$$

Using the partitioning of a matrix,

$$\begin{bmatrix} \mathbf{1} & \mathbf{a_0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \quad \ldots(6{\cdot}7)$$

where $\mathbf{a_0} = (a_{01}, a_{02}, \ldots, a_{0n}, \ldots, a_{0,n+m})$ and the remaining symbols have their usual meanings.

The matrix equation (6·7) can be expressed in the original notation form as

$$\begin{bmatrix} \mathbf{1} & -\mathbf{C} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix} \quad \ldots(6{\cdot}7)'$$

Equation (6·7) or (6·7)′ is referred to as ***standard form–I*** for the revised simplex method.

6.4. NOTATIONS FOR STANDARD FORM–I

It has been observed earlier that all the vectors have $(m+1)$ components instead of m. Hence superscript$^{(1)}$ is used for all vectors to show that they have $(m+1)$ components in standard form–I.

(I) Corresponding to each $\mathbf{a_j}$ in $\mathbf{A}$, a new $(m+1)$-component vector is represented by $\mathbf{a_j}^{(1)}$ as :

$$\mathbf{a_j}^{(1)} = [-c_j, a_{1j}, a_{2j}, \ldots, a_{mj}], j = 1, 2, \ldots, n+m$$

or
$$\mathbf{a_j}^{(1)} = [a_{0j}, a_{1j}, \ldots, a_{mj}], j = 1, 2, \ldots, n+m$$

or
$$\mathbf{a_j}^{(1)} = [a_{0j}, \mathbf{a_j}]. \quad \ldots(6{\cdot}8)$$

(II) Similarly, corresponding to m-component vector $\mathbf{b}$ in $\mathbf{AX} = \mathbf{b}$, we shall represent the $(m+1)$ component vector by $\mathbf{b}^{(1)}$ given by

$$\mathbf{b}^{(1)} = [0, b_1, b_2 \ldots, b_m] = [0, \mathbf{b}] \quad \ldots(6{\cdot}9)$$

(III) The column vector corresponding to z (or x_0) is the $(m+1)$ component unit vector which is usually denoted by $\mathbf{e_1}$ and will always be in the first column of the basis matrix $\mathbf{B_1}$ (the subscript 1 will show that it is of order $(m+1)\times(m+1)$) whose remaining m columns are any $\mathbf{a}_j^{(1)}$ such that the corresponding $\mathbf{a}_j$ are linearly independent and denoted by $\beta_i^{(1)}$, $i=1, 2, \dots, m$ (in some order).

Therefore, $\mathbf{B}_1 = [\mathbf{e}_1, \beta_1^{(1)}, \dots, \beta_m^{(1)}] = [\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)}]$ ...(6·10)

If the basis matrix **B** for $\mathbf{AX} = \mathbf{b}$ be represented by

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2m} \\ \dots & \dots & \dots & \dots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mm} \end{bmatrix},$$

then, from equation (6·10),

$$\mathbf{B}_1 = \begin{array}{c} \begin{matrix} e_1 & \beta_1^{(1)} & \beta_2^{(1)} & \dots & \beta_m^{(1)} \end{matrix} \\ \begin{bmatrix} 1 & : & -c_{B1} & -c_{B2} & \dots & -c_{Bm} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & : & \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ 0 & : & \beta_{21} & \beta_{22} & \dots & \beta_{2m} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & : & \beta_{m1} & \beta_{m2} & \dots & \beta_{mm} \end{bmatrix} \end{array} \quad \dots(6·11)$$

where $-c_{Bi}$ ($i = 1, 2, \dots, m$) are the coefficients of x_{Bi} ($i = 1, 2, \dots, m$) in the equations

$z - c_1x_1 - c_2x_2 - \dots - c_nx_n - 0x_{n+1} - \dots - 0x_{n+m} = 0$, and $\mathbf{C_B} = [c_{B1}, c_{B2}, \dots, c_{Bm}]$.

Thus, the basis matrix $\mathbf{B}_1$ [in equation (6·11)] can be represented in the partitioned form as

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{1} & -\mathbf{C_B} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \quad \dots(6·12)$$

Now the right side of (6·12) can be frequently used to obtain the basis matrix $\mathbf{B}_1$ in revised simplex method for *standard form–I*.

(IV) To compute $\mathbf{B}_1^{-1}$.

Since it is very essential to find $\mathbf{B}_1^{-1}$, compute this by applying the following rule of matrix algebra.

If
$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{Q} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}, \quad \dots(6·13)$$

where $\mathbf{R}^{-1}$ exists and is known, then inverse of matrix **M** is computed by the formula

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{QR}^{-1} \\ \mathbf{0} & \mathbf{R}^{-1} \end{bmatrix}. \quad \dots(6·14)$$

Now, to apply this rule to compute $\mathbf{B}_1^{-1}$, compare the matrices $\mathbf{B}_1$ (6·12) and **M** (6·13) to get

$$\mathbf{I} = [\mathbf{1}], \quad \mathbf{Q} = -\mathbf{C_B} \text{ and } \mathbf{R} = \mathbf{B}.$$

Substituting these values of **I**, **Q**, **R** in the formula (6·14) for matrix inverse, we get

$$\mathbf{B}_1^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{C_B B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix}. \quad \dots(6·15)$$

(V) Any $\mathbf{a}_j^{(1)}$ (not in the basis matrix $\mathbf{B}_1$) can be expressed as the linear combination of column vectors

$$(\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)})$$

in $\mathbf{B}_1$. Therefore,

$\mathbf{a}_j^{(1)} = x_{0j}\,\beta_0^{(1)} + x_{1j}\,\beta_1^{(1)} + \dots + x_{mj}\,\beta_m^{(1)} = (x_{0j}, x_{1j}, \dots, x_{mj})\,(\beta_0^{(1)}, \beta_1^{(1)}, \dots, \beta_m^{(1)}) = \mathbf{X}_j^{(1)}\,\mathbf{B}_1$, [from (6·10)]

which yields $\mathbf{X}_j^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_j^{(1)}$. ....(6·16)

(VI) A very interesting result can be obtained by using the formula (6·15) and (6·16). Substituting $\mathbf{B}_1^{-1}$ from (6·15) in (6·16),

$$\mathbf{X}_j^{(1)} = \begin{bmatrix} \mathbf{1} & \mathbf{C_B B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} -c_j \\ \mathbf{a_j} \end{bmatrix} = \begin{bmatrix} -c_j + \mathbf{C_B B}^{-1}\,\mathbf{a_j} \\ 0 + \mathbf{B}^{-1}\,\mathbf{a_j} \end{bmatrix} = \begin{bmatrix} -c_j + z_j \\ \mathbf{X_j} \end{bmatrix} = \begin{bmatrix} z_j - c_j \\ \mathbf{X_j} \end{bmatrix} = \begin{bmatrix} \Delta_j \\ \mathbf{X_j} \end{bmatrix} \quad \dots(6·17)$$

It is interesting to note from result (6·17) that the first component of $\mathbf{x}_j^{(1)}$ is $(z_j - c_j)$ or (Δ_j), which is always used to decide the optimality.

Note. The greatest advantage of treating the objective function as one of the constraints is that, $z_j - c_j$ or (Δ_j) for any $\mathbf{a}_j$ not in the basis can be easily computed by taking the product of first row of $\mathbf{B}_1^{-1}$, with $\mathbf{a}_j^{(1)}$ not in the basis, that is,

$\Delta_j = z_j - c_j =$ (first row of $\mathbf{B}_1^{-1}$) $\times\ \mathbf{a}_j^{(1)}$ not in the basis.

(VII) The $(m+1)$-component solution vector $\mathbf{X}_\mathbf{B}^{(1)}$ is given by

$$\mathbf{X}_\mathbf{B}^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{b}^{(1)} \qquad \text{...(6·18)}$$

or

$$\mathbf{X}_\mathbf{B}^{(1)} = \begin{bmatrix} 1 & \mathbf{C_B B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 \times 0 + \mathbf{C_B}(\mathbf{B}^{-1}\mathbf{b}) \\ \mathbf{0} \times 0 + \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{C_B X_B} \\ \mathbf{X_B} \end{bmatrix} = \begin{bmatrix} z \\ \mathbf{X_B} \end{bmatrix} \qquad [\text{because } \mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b},\ \mathbf{C_B X_B} = z]$$

Thus,

$$\mathbf{X}_\mathbf{B}^{(1)} = \begin{bmatrix} \mathbf{C_B X_B} \\ \mathbf{X_B} \end{bmatrix} = \begin{bmatrix} z \\ \mathbf{X_B} \end{bmatrix} \quad \textbf{(Note)} \qquad \text{...(6·19)}$$

In (6·19), it has been observed that $\mathbf{X}_\mathbf{B}^{(1)}$ is a basic solution (not necessarily feasible, because z may be negative also) for the matrix equation (6·7)′ corresponding to the basis matrix $\mathbf{B}_1$. Also, the first component of $\mathbf{X}_\mathbf{B}^{(1)}$ immediately gives the value of the objective function while the second component $\mathbf{X_B}$ gives exactly the basic feasible solution to original constraint system $\mathbf{AX} = \mathbf{b}$ corresponding to its basis matrix $\mathbf{B}$. Thus the result (6·19) is of great importance.

Now the results of this section are applied for computational procedure of revised simplex method.

6.5. TO OBTAIN INVERSE OF INITIAL BASIS MATRIX AND INITIAL BASIC FEASIBLE SOLUTION

6.5·1. When No Artificial Variables are Needed

As discussed in **section 6·4**, the inverse of initial basis matrix $\mathbf{B}_1$ is given by

$$\mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C_B B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \qquad \text{...(6·20)}$$

But, the initial basis matrix $\mathbf{B}$ for the original problem is always $(m \times m)$ identity matrix $(\mathbf{I}_m)$. It should be noted that $\mathbf{I}_m$ always appears in $(\mathbf{AX} = \mathbf{b})$ (if it is not so, it can be made to appear in $\mathbf{A}$ by introducing the artificial variables).

Since $\quad \mathbf{B} = \mathbf{I}_m = \mathbf{B}^{-1}, \quad \mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C_B I}_m \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \quad$ or $\quad \mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C_B} \\ 0 & \mathbf{I}_m \end{bmatrix}$

Furthermore, if after ensuring that all $b_i \geq 0$, only the slack variables are needed and the initial basis matrix $\mathbf{B} = \mathbf{I}_m$ appears, then

$$c_{B1} = c_{B2} = c_{B3} = \ldots = c_{Bm} = 0,\ i.e.\ \mathbf{C_B} = \mathbf{0}.$$

Thus, (6·20) becomes

$$\mathbf{B}_1^{-1} = \begin{bmatrix} 1 & : & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & : & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ : & : & & : \\ 0 & 0 & \ldots & 1 \end{bmatrix} = \mathbf{I}_{m+1}$$

Thus, it can be concluded that the inverse of the initial basis matrix $\mathbf{B}$ will be $\mathbf{B}_1^{-1} = \mathbf{B}_1 = \mathbf{I}_{m+1}$ to start with the revised simplex procedure.

Then, the initial basic solution becomes

$$\mathbf{X}_\mathbf{B}^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{b}^{(1)} = \mathbf{I}_{m+1}\,\mathbf{b}^{(1)} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \qquad \text{...(6·21)}$$

which is feasible.

$\therefore \qquad \Delta_k = \Delta_1 \Rightarrow k = 1.$

Hence $\mathbf{a}_1^{(1)}$ enters the basis. This indicates that the corresponding variable x_1 will enter the solution.

Now, in order to find the leaving vector in ***Step 7,*** first compute $\mathbf{X}_k^{(1)}$ for $k = 1$ in the next step.

Step 6. ***Compute column vector*** $\mathbf{X}_k^{(1)}$ ***(for k = 1).***

Since $\mathbf{X}_k^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_k^{(1)} = \mathbf{I}_{m+1}\,\mathbf{a}_k^{(1)}$ therefore, $\mathbf{X}_1^{(1)} \equiv \mathbf{a}_1^{(1)} = (-2, 3, 6)$.

Now complete the last column $\mathbf{X}_k^{(1)}$ of starting *Table 6·2* by writing $\mathbf{X}_1^{(1)} = \mathbf{a}_1^{(1)} = (-2, 3, 6)$ in that column. So the starting Table 6·2 grows to the following form.

Table 6·3

Variables in the basis	$\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $\mathbf{a}_3^{(1)}$	$\beta_2^{(1)}$ $\mathbf{a}_4^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$
z	1	0	0	0	−2
x_3	0	1	0	6	3
x_4	0	0	1	3	6

Step 7. ***Determination of the leaving vector*** $\beta_r^{(1)}$***, given the entering vector*** $\mathbf{a}_1^{(1)}$.

The vector $\beta_r^{(1)}$ to be removed from the basis is determined by using the **minimum ratio rule** (similar to that of ordinary simplex method) to find the value of suffix r for predetermined value of k (= 1) . *i.e.*,

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0 \text{ for } k = 1\right] = \min_i \left[\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right] = \min\left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{6}{3}, \frac{3}{6}\right] = \frac{3}{6}.$$

$$\therefore \quad \frac{x_{Br}}{x_{r1}} = \frac{x_{B2}}{x_{21}} \Rightarrow r = 2 \text{ (Equating the suffixes on both sides } ({}_{r1} = {}_{21}) \text{ find } r = 2 .)$$

The value of r thus obtained shows that the vector $\beta_2^{(1)}$ must leave the basis.

Table 6·4

Variables in the basis	$\mathbf{e}_1$	$\beta_1^{(1)}$ $(a_3^{(1)})$	$\beta_2^{(1)}$ $(a_4^{(1)})$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$	Min. ratio rule : min. $\left(\frac{\mathbf{X}_B}{\mathbf{X}_1}\right)$
z	1	0	0	0	−2	
$x_{B1} = x_3$	0	1	0	6	3	6/3
$x_{B2} = x_4$	0	0	1	3	[6]	3/6 ←
			↓ Leaving vector $\beta_2^{(1)}$		↑ Key column	

Note. It is interesting to note that the entire process of *Step 7* can be more conveniently performed by adding one more column in *Table 6·3*, for 'minimum ratio rule' (as we have seen in ordinary simplex method). *In table 6.4,* we observe that the number 6 in the column $\mathbf{X}_1^{(1)}$ comes out to be the *'key element* or *pivot element*. So we must bring unity at its place and zero at all other places of this column $\mathbf{X}_1^{(1)}$ in order to determine the transformed table from which the new (improved) solution can be read off.

Remark. If the $\min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0\right]$ is attained for more than one value of i, the resulting basic feasible solution will be degenerate.

So, in order to ensure that cycling will never occur, we shall use our usual techniques to resolve the degeneracy.

Step 8. ***Determination of the improved solution by transforming Table 6·4.***

In order to bring uniformity with the ordinary simplex method, adopt the simple matrix transformation rules which are easier for hand computations. Here the intermediate coefficient matrix can be written as :

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$
R_1	0	0	0	−2
R_2	1	0	6	3
R_3	0	1↓	3	[6]

[It should be remembered that the column $\mathbf{e}_1$ *will never change. So there is no need to write the column* $\mathbf{e}_1$ *in the above intermediate coefficient matrix. Also, because the vector* $\mathbf{x}_1^{(1)}$ *is going to be replaced by the outgoing vector* $\beta_2^{(1)}$*, the column* $\mathbf{x}_1^{(1)}$ *is placed outside the rectangular boundary].*

Now, divide the row R_3 by key element 6. Then add twice of third row to first, and subtract 3 times of third row from second. In this way, obtain the next matrix. Now the vector $\beta_2^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has been thrown out of the basis matrix and it has entered in place of $\mathbf{x}_1^{(1)}$. In this way, the process of entering $\mathbf{a}_1^{(1)}$ and removing $\beta_2^{(1)}$ (*i.e.*, $\mathbf{a}_4^{(1)}$) from the basis is now complete. Accordingly, write the column $\mathbf{a}_4^{(1)}$ in the *additional* table given below.

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{x}_B^{(1)}$	
0	1/3	1	0
1	− 1/2	9/2	0
0	1/6	1/2	1

Thus, the following table is obtained to start with the second iteration.

Table 6.5

Basic Var.	$\mathbf{e}_1$ (z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$ (k=2)	Min.Ratio Rule min. $(\mathbf{X}_B \backslash \mathbf{X}_2)$
z	1	0	1/3	1	−2/3	
x_3	0	1	−1/2	9/2	[7/2]	$\frac{9/2}{7/2}$ ←
→ x_1	0	0 ↓	1/6	1/2	1/6 ↑	$\frac{1/2}{1/6}$

$\mathbf{B}_1^{-1}$

Additional Table

$\mathbf{a}_4^{(1)}$	$\mathbf{a}_2^{(1)}$
0	−1
0	4
1	1

The improved solution is read from this table as :

$$z = 1\,,\ x_3 = 9/2\,,\ x_1 = 1/2\,,\ x_2 = x_4 = 0\,.$$

The last column of this table will be complete only when the further improvement in this solution is possible. This completes the first iteration. Repeat the entire procedure starting from *Step 3* to *Step 8* (if necessary) to obtain an optimum solution with a finite or infinite value of objective function.

Second Iteration

Step 9. ***Computation of*** Δ_j ***for*** $\mathbf{a}_4^{(1)}$ ***and*** $\mathbf{a}_2^{(1)}$ *,i.e.* $(\Delta_4\,, \Delta_2)$.

$$\{\Delta_4\,,\Delta_2\} = (\text{first row of } \mathbf{B}_1^{-1})\,(\mathbf{a}_4^{(1)}, \mathbf{a}_2^{(1)}) = (1\,, 0\,, \tfrac{1}{3}) \begin{bmatrix} 0 & -1 \\ 0 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\times 0 + 0\times 0 + \frac{1}{3}\times 1 \\ 1\times(-1) + 0\times 4 + \frac{1}{3}\times 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}.$$

Thus, we get $\Delta_4 = \frac{1}{3}\,,\ \Delta_2 = -\frac{2}{3}$. Since Δ_2 is still negative, the solution under test can be further improved.

Step 10. ***Determination of the entering vector*** $\mathbf{a}_k^{(1)}$.

To find the value of k, we have $\Delta_k = \min\,[\Delta_4\,, \Delta_2] = \min\,[\frac{1}{3}\,, -\frac{2}{3}] = \Delta_2$. Hence $k = 2$.

So $\mathbf{a}_2^{(1)}$ should enter the solution, means that the variable x_2 will enter the basic solution.

Step 11. ***Determination of the leaving vector, given the entering vector*** $\mathbf{a}_2^{(1)}$.

Compute the vector $\mathbf{X}_2^{(1)}$ so that the column $\mathbf{X}_k^{(1)}$ for $k = 2$ in *Table 6·5* may be complete at this stage.

$$\mathbf{X}_2^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_2^{(1)} = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+0+1/3 \\ 0+4+-1/2 \\ 0+0+1/6 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 7/2 \\ 1/6 \end{bmatrix}.$$

Now, instead of preparing a fresh table for performing necessary steps in second iteration, increase one more column for '*minimum ratio rule*' in Table 6·5 (which is the last table of first iteration).

The '*minimum ratio rule*' shows that 7/2 is the key element.

So remove the vector $\beta_1^{(1)}$ from the basis, to bring it in place of $\mathbf{X}_2^{(1)}$ by matrix transformation.

Step 12. ***Determination of new table for improved solution.***

For this, the intermediate coefficient matrix is :

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_2^{(1)}$
R_1	0	1/2	1	– 2/3
R_2	1	– 1/2	9/2	[7/2]
R_3	0	1/6	1/2	1/6
	↓			↑

Applying the operations : $R_2 \to \frac{2}{7} R_2$, $R_1 \to R_1 + \frac{2}{3}\left(\frac{2}{7} R_2\right)$, and $R_3 \to R_3 - \frac{1}{6}\left(\frac{2}{7} R_2\right)$, we get

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	
4/21	5/21	13/7	0
2/7	– 1/7	9/7	1
– 1/21	8/42	2/7	0

Now, the table for improved solution is as follows :

Table 6.6

Variables in the basis	z $\mathbf{e}_1$	$\mathbf{X}_2^{(1)}$ $\beta_1^{(1)}$	$\mathbf{X}_1^{(1)}$ $\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
z	1	4/21	5/21	13/7	
$x_2 = x_{B1}$	0	2/7	–1/7	9/7	
$x_1 = x_{B2}$	0	–1/121	4/21	2/7	

$\mathbf{B}_1^{-1}$

Additional Table

$\mathbf{a}_4^{(1)}$	$\mathbf{a}_3^{(1)}$
0	0
0	1
1	0

The improved solution is : $z = 13/7$, $x_2 = 9/7$, $x_1 = 2/7$.

Third Iteration

Step 13. ***Computation of Δ_4 for $\mathbf{a}_4^{(1)}$ and Δ_3 for $\mathbf{a}_3^{(1)}$.***

$$\{\Delta_4 , \Delta_3\} = (\text{first row of } \mathbf{B}_1^{-1})\ (\mathbf{a}_4^{(1)} , \mathbf{a}_3^{(1)}) = (1, 4/21 , 5/21) \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or
$$\{\Delta_4, \Delta_1\} = \begin{bmatrix} 1 \times 0 + 4/21 \times 0 + 5/21 \times 1 \\ 1 \times 0 + 4/21 \times 1 + 5/21 \times 0 \end{bmatrix} = \begin{bmatrix} 5/21 \\ 4/21 \end{bmatrix} \quad \therefore \ \Delta_4 = 5/21 \ ; \Delta_3 = 4/21 \ .$$

The positive values of Δ_4 and Δ_3 indicate that the optimal solution is : $z = 13/7$, $x_2 = 9/7$, $x_1 = 2/7$.

Remark. While solving the numerical problems by revised simplex method, the students need not give full explanation of each step. Here, we have given the detailed explanation of each step, so that the students may be able to follow each step correctly.

6.7. MORE EXAMPLES ON STANDARD FORM–I

Example 2. *Solve the following problem by revised simplex method :*

$$Max.\ z = x_1 + 2x_2 \text{, subject to}$$

$$x_1 + x_2 \le 3,\ x_1 + 2x_2 \le 5,\ 3x_1 + x_2 \le 6, \text{ and } x_1 , x_2 \ge 0 \ .$$

[Garhwal 97; Meerut M.Sc. (L.P.) 94; 90; (B.A. Pvt.) 90; Gauhati (M.C.A.) 92]

Solution. First express the given problem in revised simplex form :

$$\begin{aligned} z - x_1 - 2x_2 \qquad\qquad &= 0 \\ x_1 + x_2 + x_3 \qquad &= 3 \\ x_1 + 2x_2 \quad + x_4 \quad &= 5 \\ 3x_1 + x_2 \qquad\quad + x_5 &= 6 \ . \end{aligned}$$